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Cosmic Strings in Black Hole Spacetimes

By
Shaun Hendy ©

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Abstract

We study stationary cosmic strings in black hole spacetimes. The Kerr spacetime is stationary and axisymmetric, possessing two Killing vectors $\xi_{(t)}$ and $\xi_{(\phi)}$. A rigidly rotating string is defined as a string whose world-sheet has a tangent vector $\xi_{(t)}+\Omega\xi_{(\phi)}$ for some constant Ω. Using this rigid rotation ansatz, we solve the Nambu-Goto equations of motion for test strings in the Kerr spacetime. We classify the solutions that arise from this ansatz in the equatorial plane of the Kerr spacetime. When $\Omega = 0$, there is a family of infinitely long strings that pass throught the static limit of the Kerr spacetime in a regular way. These solutions are remarkable because they have the induced geometry of a 2-dimensional black hole. They are also the only regular and stationary test string solutions which pass through the static limit. We study the properties of these solutions and show that there are similar solutions in a wide class of algebraically special spacetimes.

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Chapter 1

Introduction

Modern theories suggest that the universe underwent a series of phase transitions in the first few fractions of a second after its birth. Like the more familiar phase transitions of condensed matter systems, these may have led to the formation of defects such as strings and monopoles, among others. Such defects are topologically stable and a very few may even have survived until the present day [1]. Cosmic strings have been studied extensively due to the role they may have played in the formation of galaxies and clusters of galaxies.

Cosmic strings arise in gauge theories with spontaneously broken symmetries (Higgs models, for example). If the symmetry groups satisfy certain topological conditions, the field equations of the theory admit stable string-like solutions with the quantized flux of a massive gauge field running along each string. In theories that also admit monopoles, the strings can end on these monopoles (the magnetic flux of the monopole is confined to the string). Otherwise the strings are either infinite or form closed loops. We will be more concerned with the latter two cases. For recent reviews on the formation of topological defects see [1], [2].

There are a number of ways one may hope to detect cosmic strings. It may be possible to detect them directly by observing their decay products [1]. Alternatively one might hope to observe the gravitational interactions of a cosmic string or network of cosmic strings. Such a network should have left behind characteristic footprints as small perturbations of the cosmic microwave background (for recent constraints see [3]). Pulsar timing observations currently provide a method for bounding the size of

these perturbations. It may also be possible to detect their gravitational interactions with other astrophysically massive objects with the new generation of gravitational wave detectors which are set to revolutionize astronomy over the next few decades [3].

Our goal is to move towards a better understanding of the latter possibility for detection. In this thesis we will examine certain aspects of the gravitational interaction of cosmic strings with black holes. Very large black holes are thought to exist at the center of galaxies, and in a close encounter with a cosmic string, such a black hole can capture the string. As the captured string interacts with the black hole, large amounts of gravitational radiation can be emitted.

We will be concerned with the final stationary states of cosmic strings trapped by a black hole. Using a stationary string ansatz for Nambu-Goto strings we will solve the equations of motion for test strings in the Kerr and Kerr-Newman spacetimes. This provides us with a family of new rigidly rotating string solutions in the Kerr spacetime. We will investigate and classify this family of solutions. One intriguing sub-family are the non-rotating stationary strings in the Kerr-Newman spacetime: these strings have the geometry of two-dimensional black holes. We will study these solutions as two-dimensional black holes and investigate the properties of the Kerr-Newman spacetime which lead to this remarkable behaviour.

This introduction is organised as follows: we begin by studying how and when strings arise in field theories, first in a simple Abelian Higgs theory and then in Grand Unified Theories. We then briefly discuss some simple physics of cosmic strings. Finally we offer an overview of the dissertation. In the introduction we use units where $\hbar = c = 1$.

1.1 Grand Unified Theories and Spontaneous Symmetry Breaking

The study of cosmic strings is especially interesting because it bridges the gap between the physics of the very small and the very large. Cosmic strings are topologically stable phenomena that are predicted by certain Grand Unified Theories of particle physics. They have also arisen in cosmology as plausible agents in galactic structure formation [4], [5].

The idea that physical laws should be invariant under groups of transformations (symmetry groups) is a very powerful one. Particle physicists today are driven by the search for a single underlying symmetry that would relate all the interactions of particles and fields. Today most theories in particle physics are expressed as what are called quantum field theories (QFTs). These theories essentially consist of a set of fields, a set of symmetries and a Lagrangian which is invariant under these symmetries. A good review of symmetry in quantum field theory is [6].

The most powerful type of symmetry group is the gauge or local symmetry where the Lagrangian is invariant under a symmetry transformation that can vary from point to point in spacetime. Such symmetries are always associated with a gauge field. For example gravity can be formulated as a gauge theory of local Lorentz transformations: in this case the gauge field is the gravitational field. Symmetry transformations which do not vary from point to point in spacetime are called global symmetries.

In practise, the fields of a QFT have well-defined transformations under the symmetry groups of the theory; i.e. they form representations of the symmetry groups. There are two types of symmetries which arise in QFTs:

- spacetime symmetries which come from the symmetries of the background spacetime. For example, in flat spacetime the spacetime symmetry group is the Poincare group (translations, rotations and Lorentz boosts). The representations of the Poincare group are labelled by mass and spin.

- internal symmetries which transform the fields into one another. QFTs possess a finite number of fields so internal symmetry groups are finite compact groups (either discrete or Lie groups). Internal gauge symmetries come with massless spin 1 gauge fields (these are often called Nambu-Goldstone bosons).

While the Lagrangian is invariant under the full symmetry group of the theory, the states of the fields, in general, will not be. Good examples of this can be found in condensed matter systems. The free energy-density of a nematic liquid crystal, which consists of rod-like or disc-like molecules, is invariant under spatial rotations [7]. The thermodynamics of the system do not depend on the orientation of the material (this is an $SO(3)$ invariance). However, at low temperatures and high pressures, the molecules tend to line up. Such states will not exhibit the full rotational symmetry but will still retain a symmetry of rotation about the axis of orientation (an $O(2)$ invariance). In this case the symmetry of the theory is said to be hidden or (more frequently) broken. The full $SO(3)$ symmetry of the theory is hidden and only the reduced $O(2)$ symmetry of the particular state is apparent.

Spontaneous symmetry breaking is a very important feature of gauge theories. There is a theorem for gauge theories with a symmetric Lagrangian and a symmetric groundstate which says that the associated spin 1 gauge boson is massless [8]. The photon is the only known massless spin 1 boson (it is the gauge boson of the electromagnetic field). Thus if gauge theories are to describe other interactions in nature the absence of the associated massless gauge bosons must be explained. The problem is resolved in the breaking of symmetry of the ground state by the Higgs

mechanism. One adds a number of scalar fields ϕ to the theory which transform non-trivially under the full symmetry group G. The potential energy density of these fields is constructed so that it has a minimum at some non-zero value ϕ_0. The ground state of the theory is then only invariant under the subgroup H of G that leaves ϕ_0 unchanged. The symmetry of G is broken to H.

It is now very well established that the electromagnetic and weak interactions can be described by a unified gauge theory based on the gauge group $SU(2) \times U(1)$. At low energies the two interactions appear very different, but the more fundamental underlying symmetry emerges at an energy scale of about 100 GeV. The theory exhibits a phase transition at a temperature of this order; at lower temperatures the theory predicts that the symmetry is broken by the Higgs mechanism.

The strong interactions are also described by a gauge theory, quantum chromodynamics, with gauge group $SU(3)$. Thus the low-energy physics of elementary particles is described by a gauge theory with three coupling constants g_3, g_2 and g_1 associated with the three groups in the low energy symmetry group $SU(3) \times SU(2) \times U(1)$. The coupling constants depend logarithmically on the energy and extrapolating from their low energy behaviour it seems probable that all three become approximately equal at an energy scale of around $10^{15} - 10^{16}$ GeV. This suggests that all three interactions may be contained in a grand unified theory which has a symmetry manifest only above this characteristic energy scale.

It is thought that there are several grand unification phase transitions. In simple models with only a single transition, the extrapolated coupling constants do not quite meet. A better fit can be obtained in models with more than one transition. Thus there emerges a picture of a sequence of phase transitions in the very early universe (the grand unification transitions would occur between 10^{-39} and 10^{-37} s). As we will see in Section 1.3, a number of grand unification schemes possess stable

cosmic string solutions which would be formed in these phase transitions. Strings are typically characterized by one dimensional parameter (as we will see below): the energy scale η (or temperature $T_c = \eta/k$) of the associated phase transition when the strings form.

An important number in the theory is the dimensionless quantity $G\mu \sim (T_c/M_{\rm Pl})^2$ which characterizes the strength of the gravitational interaction of strings. Dimensionality tells us that the mass per unit length $\mu \sim \eta^2$. Strings formed at the GUT scale (at say 10^{16} GeV) have a mass per unit length of $\mu \sim 10^{22}$ g/cm. GUT scale strings appear most commonly in the literature because the quantity $G\mu$ for the GUT scale is 10^{-6} which is the size of perturbation needed to seed galaxy formation. GUT scale strings also perturb the Cosmic Microwave Background (CMB), although recent pulsar timing measurements are placing tighter and tighter bounds on $G\mu$ (for current constraints see [3]). These CMB anisotropy studies may eventually rule out GUT scale strings.

Inflation is a generally accepted feature of almost all theories of the very early universe [5]. Inflation is a very early period of rapid expansion where the energy density is dominated by a 'vacuum energy'. Inflation has a number of appealing consequences. In particular, inflation dilutes the density of any previously existing monopoles predicted by almost all GUT theories. However, inflation also dilutes other topological defects, including cosmic strings. It is possible to have strings form very late in the inflationary epoch so that they are not inflated away [4].

In the next section we will look at a simple theory that possesses stable string solutions and examine how the strings arise because of the symmetry breaking that occurs in this theory.

1.2 The Nielson-Olesen Vortex

Consider the $U(1)$-gauge theory of a complex scalar field ϕ, the Abelian-Higgs model, in two dimensions (we follow Preskill [9] here). The theory has a gauge field A_μ and a $U(1)$-invariant Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\phi D^\mu\phi^\dagger - V(\phi) \tag{1.1}$$

where

$$D_\mu = \partial_\mu + ieA_\mu, \tag{1.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.3}$$

$$V(\phi) = \frac{1}{2}\lambda\left(|\phi|^2 - \eta^2/2\right)^2. \tag{1.4}$$

The fields transform under a $U(1)$ transformation as follows:

$$\phi \rightarrow \phi e^{i\Lambda(x^\mu)}, \tag{1.5}$$

$$A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\Lambda(x^\mu) \tag{1.6}$$

where Λ is a real-valued function of the spacetime coordinates x^μ. One can explicitly verify that (1.5-1.6) leave the Lagrangian unchanged.

The ground state (or vacuum) solution of the scalar field $\phi = \eta e^{i\Lambda_0}$ is not invariant under the transformations (1.5 - 1.6) thus the $U(1)$ symmetry is broken. The mass of the scalar field in this vacuum is $m_s = \sqrt{\lambda}\eta$. The Nambu-Goldstone boson is incorporated into the vector field A_μ which gains a mass $m_v = e\eta$ (this is the Higgs mechanism for putting massive gauge fields into a gauge field theory).

Consider a static finite-energy solution of the theory in two spatial dimensions. The energy of the field configuration is given by

$$E = \int d^2r\left(1/4F_{ij}F^{ij} + D_i\phi D^i\phi^\dagger + V(\phi)\right). \tag{1.7}$$

Each term in (1.7) must be finite if the energy is to be finite. In particular the potential $V(\phi)$ must vanish at spatial infinity for the third term to be finite. Thus the limit

$$\lim_{r\to\infty} \phi(r,\theta) = \phi(\infty,\theta) \tag{1.8}$$

(where θ is the polar angle) must exist and be a zero of the potential energy $V(\phi)$ i.e.

$$\phi(\infty,\theta) = \eta e^{i\Lambda(\theta)}/\sqrt{2}. \tag{1.9}$$

Because of this degeneracy in the vacuum states it is possible for $\phi(\infty,\theta)$ to depend non-trivially upon θ (i.e. the phase $e^{i\Lambda(\theta)}$ is not fixed by the requirement that the energy be finite).

The set of all vacuum states of the scalar field ϕ, $\left\{\phi : |\phi| = \eta/\sqrt{2}\right\}$, is topologically a 1-sphere. This space of possible vacuum states is called the vacuum manifold. Thus the vacuum expectation value of the scalar field at infinity determines a path on the vacuum manifold via the equation:

$$\phi(\infty,\theta) = \Lambda(\theta)\phi(\infty,0) \tag{1.10}$$

where $\Lambda \in U(1)$. The group elements $\Lambda(\theta)$ determine a path on the vacuum manifold parameterized by θ. Another way of saying this is that each path on the manifold is a map from the unit circle (parameterized by the polar angle θ) to the vacuum manifold.

It is here that the topology of the vacuum manifold becomes important. Paths on the vacuum manifold that can be continuously deformed into one another (i.e. deformed into one another without cutting or breaking the paths) are said to be homotopically equivalent. A path that can be deformed to the constant map $\phi_\infty(\theta) = \phi_\infty(0)$ (which corresponds to just a point on the vacuum manifold) is homotopically equivalent to a purely vacuum field configuration.

Mathematically, paths on manifolds are put into homotopic equivalence classes depending on whether or not they can be deformed into one another. Together with rules for combining two paths to make a third, the set of homotopy classes forms a group structure called the first homotopy group. In this case (where the gauge group is $U(1)$) the homotopy classes are characterized by the winding number n:

$$n = \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{d\Lambda}{d\theta}. \tag{1.11}$$

This is an integer which counts the number of times the path in field space is wound around the circle in ordinary space. The winding number n is called a topological index. It is not possible to continuously change a solution with a winding number n into a solution with a different winding number, as they are not homotopically equivalent (one cannot continuously change one integer into another). Since time evolution is continuous, the winding number is a constant of the motion. Thus the winding number is conserved; this is a topological conservation law.

We must verify that a field configuration with non-zero winding number can really have finite energy. The second term of (1.7) is dangerous because it involves a derivative of ϕ; the gradient of ϕ in the circumferential direction is non-zero at spatial infinity as $\phi_\infty(\theta)$ is a non-trivial function of θ when $n \neq 0$. The gradient term

$$\int d^2r \left| \left(\frac{1}{r}\frac{\partial}{\partial\theta} - ieA_\theta \right) \phi \right|^2 \tag{1.12}$$

is only finite if the gauge field [1] behaves for large r as

$$A_\theta \to \frac{1}{er}\frac{d\Lambda}{d\theta} + \cdots \quad \text{as } r \to \infty. \tag{1.13}$$

The gauge field (1.13) is pure gauge (i.e. there is a gauge transformation that sets $A_\theta = 0$: see equation (1.6)). Thus the prescribed asymptotic behaviour of A_θ also means that the field strength $F_{\mu\nu}$ will decay sufficiently rapidly to allow the first term

[1] If the symmetry is not local then there is no gauge field A to keep the energy finite; strings that arise from such global symmetries do not have finite energy per unit length [4].

in (1.7) to be finite. Thus it is possible for a field configuration with $n \neq 0$ to have finite energy.

The gauge field cannot be pure gauge everywhere if $n \neq 0$. There is a magnetic flux through the plane which can be found using Stokes Theorem:

$$\int_{S_\infty} r d\theta A_\theta = \frac{2\pi}{e} n. \tag{1.14}$$

where S_∞ denotes the circle at infinity. Note that the flux is quantized and the number of flux quanta is the winding number.

Further, the fact that $n \neq 0$ guarantees that the field ϕ vanishes somewhere. If ϕ has no zeroes then the phase Λ is well-defined everywhere. By smoothly shrinking the circle at infinity to an infinitesimal circle at the origin, we can smoothly deform the mapping $\Lambda(\theta)$ to the constant mapping $\Lambda = \text{const.}$ which has $n = 0$. This is not possible if the mapping $\Lambda(\theta)$ has $n \neq 0$. Thus there is at least one point where Λ is ill-defined because ϕ vanishes.

From the potential (1.4) we see that the field has energy at $\phi = 0$, i.e. there is a lump of energy surrounding this point. This lump is known as the Nielson-Olesen vortex [10]. If we consider a cyclindrically symmetric field configuration in three dimensions, which looks like the Nielson-Olesen vortex in the plane, then we have a line of finite energy density at $\phi = 0$, i.e. a string. In other words, the Nielson-Olesen vortex can be thought of as the cross-section of an infinite string.

However, the vortex is not necessarily energetically stable for $n > 1$. There are two characteristic length scales for the vortex solution. The first length scale, say $r_s \sim m_s^{-1}$, is the radius of the region where the field $\phi(r, \theta)$ departs significantly from its vacuum state; semiclassically this is the Compton wavelength of scalar particle. The second is the radius, say $r_v \sim m_v^{-1}$, of the region where the field A_θ departs significantly from its vacuum state; again, this corresponds to the Compton wavelength

of the vector particle.

The gauge field generates a repulsive force because the quanta of magnetic flux (1.14) repel each other. The scalar field generates an attractive force because the scalar field prefers to be localized (deviations from the vacuum state cost potential energy). If $m_s > m_v$ the scalar boson is heavier, the gauge force dominates and the vortices repel; a flux $2\pi n/e$ vortex with $n > 1$ will split into n vortices carrying unit flux $2\pi/e$. If $m_v > m_s$ then the vortices are stable for any n (note that $m_v/m_s = \lambda/e^2$).

We can estimate the mass of a stable vortex with $m_v > m_s$ as follows: the vortex consists of a lump of scalar field with energy density $\sim \lambda\eta^4$. Thus the mass contribution of the scalar part of the vortex is $\mu_s \sim \lambda\eta^4\delta_s^2 \sim \eta^2$. The vortex also has quanta of magnetic flux with energy density $\sim e^2\eta^4$. The vector field therefore contributes mass $\mu_v \sim e^2\eta^4\delta_v^2 \sim \eta^2$. Thus the mass of the vortex $\mu \sim \eta^2$ and the Compton wavelength of the vortex is $\delta \sim 1/\mu \sim 1/\eta^2$.

More precisely it can be shown that

$$\mu_{vortex} \simeq \pi\eta^2 \ln m_s/m_v. \tag{1.15}$$

In three dimensions the mass per unit length, μ_{string}, of the string is

$$\mu_{string} \simeq \pi\eta^2 \ln m_s/m_v. \tag{1.16}$$

Thus the linear density depends strongly on the energy of the vacuum state η and only weakly on the parameters e and λ.

1.3 Strings in Unified Theories

In the previous section, we considered an Abelian-Higgs model which for certain parameter values possessed stable string solutions. Whether a theory with a larger

symmetry group than $U(1)$ possesses stable string solutions, depends on the topology of the vacuum manifold of the theory. In this section, we briefly state some general results for non-Abelian symmetry groups.

Let's consider a theory with a general symmetry group G and vacuum manifold $\mathcal{M}$. If ϕ_0 is a point on $\mathcal{M}$ then for any $g \in G$, $g\phi_0$ is also in $\mathcal{M}$ [2]. Define the isotropy group H of ϕ_0 to be the set of all elements $h \in H$ such that $h\phi_0 = \phi_0$. Then clearly $g\phi_0 = g'\phi_0$ if and only if $g^{-1}g' \in H$ i.e. the points of $\mathcal{M}$ are in one-to-one correspondance with left cosets of H in the group G. This is often indicated by writing $\mathcal{M} = G/H$.

The argument as to whether or not the theory possesses strings runs much as in the Abelian case. Again it comes down to classifying non-contractible maps from the circle at infinity to the vacuum manifold. In the Abelian case, the classes of homotopically equivalent maps formed a group; these classes were distinguished by their integral winding number, so it is not surprising that the homotopy group in question can be shown to be the integers. In general, the group formed by the classes of homotopically equivalent maps is called the fundamental or first homotopy group and is denoted $\pi_1(\mathcal{M})$.

If $\pi_1(\mathcal{M})$ is the trivial group, $\mathcal{M}$ is said to be simply connected. A necessary, but not sufficient, condition for the existence of stable vortices is that $\pi_1(\mathcal{M})$ be a non-trivial group. If there does exist a non-contractible map (so that $\pi_1(\mathcal{M})$ is non-trivial), then we may suspect the existence of stable vortices but we cannot gaurantee this. In the Abelian case, we saw that for a winding number $n > 1$ and $m_s > m_v$ there are no stable vortices. Similar energy conditions must be taken into account for each non-Abelian theory case by case.

It is possible to show that electroweak theory does not possess topologically

[2] We will assume here that every element in $\mathcal{M}$ is of the form $g\phi_0$ to simplify the discussion.

stable string solutions because its vacuum manifold is isomorphic to a three-sphere which is simply connected. Thus if cosmic strings exist in nature, they must result from breaking symmetries which are currently unknown, perhaps those of a GUT. The minimal grand unification scheme (based on the symmetry group $SU(5)$) does not possess stable string solutions either. However, there are a number of plausible schemes that do have stable string solutions [4]. The observation (or lack of observation) of cosmic strings would be a useful discriminant among grand unification schemes.

1.4 The Physics of Cosmic Strings

We can deduce some further properties of cosmic strings simply from Lorentz invariance and dimensional arguments. Consider a straight string lying along the z-axis. It is described by a solution of Lorentz invariant field equations and therefore is invariant under boosts in the z-direction. Hence the rest frame of the string is only defined up to longitudinal boosts; only transverse motion of the string has any physical meaning.

As we learned by studying the Nielson-Olesen vortex solution of the Abelian-Higgs theory the string is characterized by the energy scale η of the symmetry breaking. The mass per unit length of the string was found to be $\mu \sim \eta^2$. Further the thickness of the string $\delta \sim \eta^{-1}$.

Formally we begin by defining the position of the string to be the location of the zeroes of the scalar field $\phi(x)$, which we denote

$$X^\mu = X^\mu(\zeta^0, \zeta^1) \tag{1.17}$$

This world history of the string corresponds to a two-dimensional timelike surface embedded in spacetime (we will refer to this two-surface as a world sheet). The ζ^A

are coordinates on the world sheet. This embedding of a two-surface in a background spacetime induces a metric on the string world sheet

$$G_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \zeta^A} \frac{\partial x^\nu}{\partial \zeta^B}. \tag{1.18}$$

To determine the dynamics of a string, we should propose an appropriate action functional for the string. This action must be invariant under general spacetime coordinate transformations and also under reparameterisations of the two-surface. For strings that arise from gauge theories, it should also be local. Thus the action must have the form

$$S = \int \mathcal{L} \sqrt{-G} d^2\zeta, \tag{1.19}$$

where $\mathcal{L}$ is some Lagrangian density.

The Lagrangian for a static solution in a flat background spacetime is just the negative of the energy density. In terms of the mass per unit length, μ, the action for a straight string on the z-axis is the Nambu-Goto action [11]:

$$\begin{aligned} S &= -\mu \int dt dz, \\ &= -\mu \int \sqrt{-G} d^2\zeta. \end{aligned} \tag{1.20}$$

This last expression is covariant in both two and four dimensions, so it holds for any background metric $g_{\mu\nu}$ and for any embedding $X^\mu(\zeta)$, providing that the string is almost straight. This is the first term in the expansion of the effective action for the string in powers of curvature. This is the action we will make use of in this thesis; we will examine the conditions under which we can use this effective action in the next chapter.

1.5 Overview

In the subsequent work we will use units where $G = c = 1$ unless otherwise noted; occasionally we will insert factors of G and c where needed. In chapters 2-4, we will use a metric with signature $(-1, 1, 1, 1)$. In chapter 5 and appendices, to make contact with existing literature, we use a metric with signature (1,-1,-1,-1).

1.5.1 Chapter Two

We will begin this chapter by presenting the mathematical background necessary for the study of cosmic string solutions in curved spacetime. The Nambu-Goto action is a good approximation to the behaviour of strings in curved spacetimes only in what is often called the test string limit. This is the limit where the curvature of the string remains small and the effects of back-reaction on spacetime due to the string can be neglected. For the purposes of this dissertation, however, studying Nambu-Goto strings will be sufficient. Consequently, we will revisit the Nambu-Goto action in detail and also consider the equivalent Polyakov action for strings. By variation of the Polyakov action, we will finally obtain equations of motion for test strings. We introduce a formalism due to Larsen and Frolov [12] for studying the propagation of perturbations on background Nambu-Goto strings. Finally, as a primer, we will look at static test string solutions in Minkowski spacetime.

1.5.2 Chapter Three

The Kerr spacetime is stationary and axisymmetric with two Killing vectors: $\xi^\mu_{(t)} = \delta^\mu_t$ and $\xi^\mu_{(\phi)} = \delta^\mu_\phi$. Note that the following combination $\chi^\mu = \xi^\mu_{(t)} + \Omega\xi^\mu_{(\phi)}$ of the Killing vectors $\xi^\mu_{(t)}$ and $\xi^\mu_{(\phi)}$ is also a Killing vector provided Ω is constant. In a region where

$\chi^\mu = (1,0,0,\Omega)$ is timelike one can define a set of Killing observers whose four-velocities are $u^\mu = \chi^\mu / |\chi^2|^{1/2}$. This set of observers form a rigidly rotating reference frame that is the frame moving with a constant angular velocity Ω.

- By performing the following coordinate transformation

$$\varphi = \phi - \Omega t \tag{1.21}$$

we can look for solutions to the Nambu-Goto equations of the form

$$x^\mu(\zeta^A) = (\tau, r(\sigma), \theta(\sigma), \varphi(\sigma)) . \tag{1.22}$$

Note that for this form of solution the Killing vector $\chi^\mu = \frac{\partial x^\mu}{\partial \tau}$ is tangent to the corresponding world sheet. Thus, in regions where χ is timelike, the form of the string will seem rigid in the frame moving with constant angular velocity Ω.

This ansatz provides us with a pair of coupled second-order ODE's in φ and θ. We will consider a few of the known solutions to these equations. In the equatorial plane of the Kerr metric, solutions are known for all values of Ω (Frolov, Hendy and De Villiers [13]). It will be shown in this chapter that solutions exist even when χ is spacelike. Since only the normal velocity of the string is physical, rigid configurations can appear to rotate superluminally while retaining a timelike normal velocity.

We conclude this section by briefly studying the gravitational radiation from trapped strings.

1.5.3 Chapter Four

In this chapter we study world sheets that are tangent to the Killing vector $\xi_{(t)}$ (so $\Omega = 0$). These were first studied by Frolov et al [14]. A particularly interesting class of solutions in this family are strings which lie on the cones $\theta = \text{const}$. These have

been investigated in detail in Frolov et al [15, 15, 16]; we will present some of this work here in chapter 4. The induced metric on the world sheet of these strings is that of a two-dimensional black hole. Hence results from the study of two-dimensional black holes can be applied to objects with more of a physical basis. For example, these two-dimensional black holes would also radiate the string perturbation analog of Hawking radiation. The equations of motion for perturbations along the string, 'stringons' were also obtained in this analysis.

A remarkable feature of these solutions is that the causality of the two-dimensional geometry on the string differs from that in the four-dimensional embedding spacetime. It is possible to send signals from inside the two-dimensional black hole to the exterior using the extra dimensions of the surrounding spacetime.

These cone strings also possess several interesting geometric properties:

- the world sheet of these strings is tangent to the principal null vectors of the Weyl tensor in the Kerr-Newman spacetime;
- the cone strings are the only stationary world sheets (with respect to the Killing vector $\xi_{(t)}$) that can pass through the static limit of the black hole (the surface where $\xi^2 = 0$ and inside which no static trajectories exist).

These properties stem from the fact that the principal null vectors of the Weyl tensor $l_{\pm}$ are geodesic and are eigenvectors of $\nabla_\nu \xi^\mu$ where ∇_ν is the covariant derivative of the Kerr spacetime;

$$l^\nu \nabla_\nu \xi^\mu = \kappa l^\mu \,. \tag{1.23}$$

Such world-sheets are labelled Principal Killing surfaces.

1.5.4 Chapter Five

It is possible to study the geometry of two-surfaces using a null tetrad formalism. We demonstrate that the Newman-Penrose formalism can be used for this purpose, and note that minimal surfaces have a simple description using these ideas. Timelike minimal surfaces are solutions of the Nambu-Goto action (1.20) and represent test string world-sheets.

We then introduce a generalization of the principal Killing surfaces which we call Principal Weyl surfaces. These surfaces are timelike minimal surfaces with a geodesic shear-free tangent vector. In a vacuum spacetime, this geodesic shear-free vector field is a repeated principal vector of the Weyl tensor. We show that these principal Weyl surfaces exist in a number of vacuum type D spacetimes. Furthermore in the Kerr and Taub-NUT spacetimes, these principal Weyl surfaces are also principal Killing surfaces so that they are tangent to a Killing vector and represent stationary string world-sheets.

An important family of algebraically special spacetimes are those which admit a Kerr-Schild metric [17]. We examine the properties of principal Weyl surfaces in these spacetimes and, in particular, ask when these surfaces are stationary.

Chapter 2

Dynamics of Cosmic Strings in Curved Spacetime

Previously we wrote down an action for cosmic strings (1.20) which was valid in the limit where the string was static and almost straight. In this limit the Nambu-Goto action determines the dynamics of cosmic strings. In this chapter we begin by introducing the mathematical machinery needed for the study of string world-sheets in curved spacetime and then return to the study the Nambu-Goto action in situations where the curvature is no longer negligible. We then study the propagation of perturbations on strings in curved spacetimes. We conclude the chapter by considering some simple examples of string solutions in Minkowski spacetime.

2.1 The Geometry of Two-Surfaces in Curved Spacetime

In this section we will define a number of geometric quantities which will be useful for the description of timelike two-surfaces embedded in a curved spacetime. Let $X^\mu = X^\mu(\zeta^A)$ define a two-surface embedded in a four-dimensional spacetime ($\mu = 0, 1, 2, 3$ and $A = 0, 1$). There is a metric G_{AB} induced on the surface by this embedding which is given by

$$G_{AB} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \zeta^A} \frac{\partial X^\nu}{\partial \zeta^B} = g_{\mu\nu} X^\mu{}_{,A} \, X^\nu_{,B}. \tag{2.1}$$

The vectors $X^\mu_{,A}$ are tangent to the two-surface. The determinant of the induced metric is denoted G.

In general two vector fields, say k^μ and l^μ, are said to be surface-forming if

$$\mathcal{L}_k l = [k,l] = k^\mu \nabla_\mu l^\nu - l^\mu \nabla_\mu k^\nu = \alpha k^\nu + \beta l^\nu \tag{2.2}$$

for some real functions $\alpha = \alpha(x)$ and $\beta = \beta(x)$ of the spacetime coordinates. Here $\mathcal{L}_k l = [k,l]$ is the Lie derivative of l in the direction of k. If this condition is satisfied in a region then one can find integral submanifolds of k and l i.e. at each point in this region we can find an embedded two-dimensional submanifold S with tangent space spanned by k and l (see Wald [18] for example). Further if $\alpha(x)$ and $\beta(x)$ both vanish identically so that

$$[k,l] = k^\mu \nabla_\mu l^\nu - l^\mu \nabla_\mu k^\nu = 0, \tag{2.3}$$

then these two-surfaces have an embedding $x^\mu(\sigma^A)$ where $k^\mu = x^\mu_{,0}$ and $l^\mu = x^\mu_{,1}$. In this case, where $[k,l] = 0$, the vector fields are said to be coordinate vector fields. Conversely, given an embedding $X^\mu = X^\mu(\zeta^A)$ as above, then the coordinate vector fields $X^\mu_{,A}$ also satisfy $[X_{,0}, X_{,1}] = 0$.

A two-surface is said to be timelike if the determinant of the induced metric $G < 0$. If the determinant of the induced metric $G > 0$ then the two-surface is said to be spacelike. We are interested in surfaces which approximate the world-history of a cosmic string and consequently we are interested in timelike surfaces. A surface is timelike if and only if there is a timelike vector tangent to that surface; in this sense, it represents the time evolution of a one-dimensional curve. It is also convenient to introduce two vectors n^μ_R (R=2,3) normal to the two-surface. For a timelike two-dimensional surface these normal vectors are spacelike and satisfy the following relations:

$$g_{\mu\nu} n^\mu_R n^\nu_S = \delta_{RS}, \qquad g_{\mu\nu} x^\mu_{,A} n^\nu_R = 0. \tag{2.4}$$

These two normal vectors span the vector space normal to the surface at a given point, and they are uniquely defined up to local rotations in the (n_2, n_3)-plane.

We will assume from now on that we are discussing timelike two-surfaces. In this case the normal vectors n^μ_R are both spacelike. This also means that G_{AB} is a Lorentzian metric. Thus the tangent vectors $x^\mu{}_{,A}$ and normal vectors n^μ_R satisfy the completeness relation:

$$g^{\mu\nu} = G^{AB} x^\mu_{,A} x^\nu_{,B} + \delta^{RS} n^\mu_R n^\nu_S. \tag{2.5}$$

We also define the second fundamental form, or extrinsic curvature, K_{RAB}. The second fundamental form is defined as:

$$K_{RAB} = g_{\mu\nu} n^\mu_R x^\rho_{,A} \nabla_\rho x^\nu_{,B}. \tag{2.6}$$

Note that K_{RAB} is symmetric in its last two indices i.e. $K_{R(AB)} = K_{RAB}$ since the commutator of the coordinate vector fields vanishes.

The second fundamental form is related to the Ricci curvature scalar $R^{(2)}$ of the two-surface by the relation:

$$R^{(2)} = K_{RAB} K^{RAB} - K_R K^R, \tag{2.7}$$

where $K^R = G_{AB} K^{RAB}$ is the trace of the second fundamental form.

The normal fundamental form is defined as

$$\mu_{RSA} = g_{\mu\nu} n^\mu_R x^\rho_{,A} \nabla_\rho n^\nu_S. \tag{2.8}$$

Since the normal vectors n^μ_R are orthogonal unit vectors the normal fundamental form is antisymmetric in its first two indices $\mu_{RSA} = \mu_{[RS]A}$.

2.2 The Equations of Motion for Cosmic Strings in Curved Spacetimes

As noted in the introduction, the location of a string can be specified by the two-surfaces in spacetime on which the scalar field $\phi(x)$ vanishes. Each particular two-surface is timelike and has a Lorentzian metric G_{AB} (2.1) induced by its embedding in the given spacetime. This metric is symmetric and so possesses three degrees of freedom. However two degrees of freedom can be removed using two-dimensional general coordinate invariance, i.e. there is always a coordinate transformation $\zeta^A \to \tilde{\zeta}^A(\zeta)$ such that

$$G_{AB} \to \tilde{G}_{AB} = \Omega^2(\tilde{\zeta})\eta_{AB} \tag{2.9}$$

where $\eta_{AB} = \text{diag}(-1, 1)$. In other words any two-dimensional surface is conformally flat. We refer to this choice of metric as the conformal gauge.

We noted in the introduction that $\mathcal{L}$ is a function of the linear mass density μ and a functional of appropriate geometric quantities (such as curvature of the world-sheet and its derivatives). We also obtained the Nambu-Goto action which was appropriate when the string was almost straight. In the case where the curvature of the string world-sheet is small but no longer negligible, we expand the action in powers of curvature. The relevant measure of curvature is the extrinsic curvature K_{RAB} defined by (2.6).

To second-order in K, the effective action takes the form (see [4]):

$$S = -\int d^2\zeta\sqrt{-G}\left(\mu - AK^R K_R + BR^{(2)}\right) \tag{2.10}$$

where A and B are dimensionless constants (recall the relationship between $R^{(2)}$ and the second fundamental form (2.7)). The integral of $R^{(2)}$ over the world-sheet can be evaluated by the Gauss-Bonnet theorem [7]: it is simply the Euler characteristic of

the world-sheet which is a topological invariant. Thus this term does not affect the equations of motion.

We will demonstrate shortly that $K^2 = K^R K_R$ vanishes identically for a solution of the Nambu-Goto equations of motion. Thus any solution of the Nambu-Goto equation is also a solution of the corrected action. To this order the second term does not affect the equations of motion.

If we are to work to this order in approximating the motion of cosmic strings we must assume that the curvature is sufficiently small. In particular we must assume that the thickness of the string is much less than the radius of curvature, R, of the string. Thus $K \sim R^{-2} << \delta^{-2} \sim 1/\mu$. Thus the Nambu-Goto action is an appropriate approximation provided $K\mu << 1$.

By solving the Nambu-Goto equations in a background spacetime we must be able to neglect the effect of back-reaction of the string's mass on the spacetime geometry so $G\mu/c^2 << 1$. For example, in the spacetime of a black hole of mass M, the motion of a string of length L in the vicinity of the black hole and linear density μ is well approximated by the Nambu-Goto action provided $\mu L << M$. If this is not the case the black hole will tend to orbit the string! Strings that satisfy these conditions are called test strings. For example, GUT strings with $G\mu \sim 10^{-6}c^2$ can, in most situations, be modelled sufficiently well as test strings.

There is another form of the action for test strings that will prove useful. Let h_{AB} be the internal metric of the world-sheet (which can be specified freely). The Polyakov action [19] is written in terms of this internal metric as follows:

$$S = \int d^2\zeta \sqrt{-h} h^{AB} G_{AB}. \tag{2.11}$$

This action for test strings is more convenient for deriving the general equations of motion; we will see that the Polyakov action is equivalent to the Nambu-Goto action.

To obtain the equations of motion we will vary the embedding X^μ and the metric h_{AB}. Under the variation δX^μ, we have the Euler-Lagrange equation

$$\partial_A \left(\sqrt{-h} h^{AB} \partial_B X_\mu\right) = 0. \tag{2.12}$$

By expanding the bracket this can be rewritten as

$$\Box X^\mu + h^{AB} \Gamma^\mu_{\rho\sigma} X^\rho_{,A} X^\sigma_{,B} = 0, \tag{2.13}$$

where $\Box$ is the d'Alembertian, given by

$$\Box = \frac{1}{\sqrt{-h}} \partial_A \left(\sqrt{-h} h^{AB} \partial_B\right) \tag{2.14}$$

and $\Gamma^\mu_{\rho\sigma} = 1/2 g^{\mu\nu}(g_{\rho\nu,\sigma} + g_{\nu\sigma,\rho} - g_{\rho\sigma,\nu})$ are the Christoffel symbols of the background spacetime.

Under the variation δh_{AB}, the integrand in (2.11) changes as

$$\begin{aligned} \delta\left(\sqrt{-h} h^{AB} G_{AB}\right) &= \delta\sqrt{-h} h^{AB} G_{AB} + \sqrt{-h} \delta h^{AB} G_{AB} \\ &= -\frac{1}{2}\sqrt{-h} h_{CD} \delta h^{CD} h^{AB} G_{AB} + \sqrt{-h} \delta h^{AB} G_{AB}. \end{aligned} \tag{2.15}$$

This is true for any variation δh_{AB}. As the two-dimensional stress-energy tensor is given by

$$T_{AB} = \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{AB}}, \tag{2.16}$$

we see that T_{AB} must vanish to extremize the action:

$$T_{AB} = G_{AB} - \frac{1}{2} h_{AB} h^{CD} G_{CD} = 0. \tag{2.17}$$

Using (2.17) to substitute for h_{AB} in the Polyakov action (2.11) it is seen that one recovers the Nambu-Goto action.

The equations of motion are then

$$G_{AB} - \frac{1}{2} h_{AB} h^{CD} G_{CD} = 0, \tag{2.18}$$

$$\Box X^\mu + h_{AB} \Gamma^\mu_{\rho\sigma} X^\rho_{,A} X^\sigma_{,B} = 0. \tag{2.19}$$

The spacetime string stress-energy tensor $T^{\mu\nu}(x)$ is given by the functional derivative of the action (2.11) with respect to $g_{\mu\nu}$:

$$\sqrt{-g}T^{\mu\nu}(x) = \frac{\delta S}{\delta g^{\mu\nu}} \tag{2.20}$$

$$= \mu \int d^2\zeta \sqrt{-h} h^{AB} \partial_A X^\mu \partial_B X^\nu \delta^{(4)}(x - X). \tag{2.21}$$

We now demonstrate as promised that the trace of the extrinsic curvature vanishes for solutions of (2.19). Contracting the normal vector n_R^μ with the d'Alembertian of X^μ gives:

$$\begin{aligned} n_{R\,\mu} \Box X^\mu &= n_{R\,\mu} \left(\frac{1}{\sqrt{-h}} \partial_A \left(\sqrt{-h} h^{AB} \partial_B \right) \right) \\ &= h^{AB} n_{R\,\mu} X^\nu_{,A} \partial_\nu X^\mu_{,B}. \end{aligned} \tag{2.22}$$

Thus, contracting $n_{R\,\mu}$ with the entire left hand side of (2.19) we find that

$$\begin{aligned} n_{R\,\mu} \left(\Box X^\mu + h_{AB} \Gamma^\mu_{\rho\sigma} X^\rho_{,A} X^\sigma_{,B} \right) &= h^{AB} n_{R\,\mu} X^\nu_{,A} \nabla_\nu X^\mu_{,B} \\ &= h^{AB} K_{RAB} \\ &= K^R. \end{aligned}$$

Thus equation (2.19) can be written compactly as

$$K^R = 0. \tag{2.23}$$

A surface for which K^R vanishes identically is called a minimal surface. Mathematically, solving the Nambu-Goto equations of motion for cosmic string solutions is equivalent to finding timelike minimal two-surfaces.

Having obtained the equations of motion (2.19) we need to discuss boundary conditions. The conservation of the magnetic flux of the massive gauge field (see Section 1.2) that runs along the string imposes constraints on the types of boundary conditions one can consider. There are two types of boundary conditions commonly

considered: infinite and closed string boundary conditions. For a closed string the embedding $X^\mu(\zeta^A)$ is periodic in a spatial coordinate so that the string forms a closed loop. An infinite string has end points at spatial infinity.

Physically there are a number of other possibilities. It is possible for composite defects to form in a series of phase transitions [1]: for example it is possible for strings to end in monopoles (which provide a source for the magnetic flux). Formally, one can consider "open" string boundary conditions, where the end-points of the string are moving at the speed of light. Such boundary conditions are, in general, not physical for cosmic strings; they do not conserve magnetic flux for example. However, fundamental strings in curved spacetime backgrounds are also studied as solutions of the Nambu-Goto action (see [20] for example). In such cases these "open" string boundary conditions are often considered. In Chapter 3 we will consider "open" string solutions as approximations to more physical string solutions. Finally, it is possible for a string solution to end on a black hole [21] because of the non-trivial topology of the black hole horizon.

2.3 Propagation of Perturbations along Strings

One can study the propagation of perturbations along the world sheet as perturbations on a background world-sheet which is an exact solution of the Nambu-Goto equations (2.18-2.19). This can be achieved by making a second variation of the Polyakov action (2.11). Here we follow Frolov and Larsen [12] (see also [22, 23, 24]). A general perturbation of the string world-sheet δX^μ can be written as

$$\delta X^\mu = \delta X^R n^\mu_R + \delta X^A X^\mu_A. \tag{2.24}$$

Note that variations of the form $\delta X^A X^\mu_A$ leave the action unchanged because of general two-dimensional coordinate invariance. Thus without loss of generality we only need

consider perturbations of the form

$$\delta X^{\mu} = \Phi^{R} n^{\mu}_{R}. \tag{2.25}$$

The second variation of the action (2.11) is quite complicated but Larsen and Frolov have performed this lengthy calculation [12]. The field δh_{AB} is not a dynamical field; the field h_{AB} does not appear in the Nambu-Goto (1.20) action and the first variation δh_{AB} of the action (2.11) leads to a constraint. Thus δh_{AB} can be eliminated from the effective action for physical perturbations. Their result gives us the effective action for physical perturbations on the background world-sheet:

$$\mathcal{S}_{\text{eff.}} = \int d^2\zeta \sqrt{-G}\, \Phi^{R} \left\{ G^{AB} (\delta^{T}_{R} \nabla_{A} + \mu_{R}{}^{T}{}_{A})(\delta_{TS} \nabla_{B} + \mu_{TSB}) + \mathcal{V}_{RS} \right\} \Phi^{S}, \tag{2.26}$$

where $\mathcal{V}_{RS} = \mathcal{V}_{(RS)}$ are scalar potentials defined as:

$$\mathcal{V}_{RS} \equiv K_{RAB} K_{S}{}^{AB} - G^{AB} x^{\mu}_{,A} x^{\nu}_{,B} R_{\mu\rho\sigma\nu} n^{\rho}_{R} n^{\sigma}_{S}. \tag{2.27}$$

The action is also invariant under local rotations of the normal vectors n^{ρ}_{R}. The equations describing the propagation of perturbations (stringons) on the world-sheet background are then found to be:

$$\left\{ \delta_{RS} \Box + 2\mu_{RS}{}^{A} \partial_{A} + \nabla_{A} \mu_{RS}{}^{A} - \mu_{R}{}^{TA} \mu_{STA} + \mathcal{V}_{RS} \right\} \Phi^{S} = 0. \tag{2.28}$$

We will make use of these equations in chapter 4 where we will study the propagation of perturbations on stationary non-rotating strings in the Kerr spacetime.

2.4 Strings In Minkowski Spacetime

Working in the conformal gauge (2.9) still allows considerable freedom. To analyse the equations of motion further, it is convenient to fix the gauge as follows: in Minkowski spacetime with metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ we identify the world sheet time $\tau = \zeta^{0}$

with Minkowski time x^0 (we also let $\sigma = \zeta^1$). This choice of gauge (the conformal gauge and this temporal gauge) imposes the constraints

$$\dot{\mathbf{X}}^2 + \mathbf{X}'^2 = 1, \qquad \dot{\mathbf{X}} \cdot \mathbf{X}' = 0. \tag{2.29}$$

where $X^\mu = (X^0, \mathbf{X})$; the dot denotes differentiation with respect to τ and the prime denotes differentiation with respect to σ. The equations of motion become

$$\mathbf{X}'' - \ddot{\mathbf{X}} = 0. \tag{2.30}$$

In this gauge we see that the string velocity $\dot{\mathbf{X}}$ is orthogonal to the string tangent vector $\mathbf{X}'$. The general solution to the wave equation (2.30) and the constraints (2.29) are

$$\mathbf{X} = \frac{1}{2}\left[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)\right], \qquad \mathbf{a}'^2 = \mathbf{b}'^2 = 1. \tag{2.31}$$

The Nambu-Goto equations are solved by specifying two curves, $\mathbf{a}'$ and $\mathbf{b}'$, on the unit sphere.

In curved spacetime, the situation is considerably more difficult. In flat spacetime the Christoffel symbols in equation (2.19) vanish; in curved spacetime the presence of the Christoffel symbols means that exact solutions can only be found in certain situations. Here we will illustrate the so-called stationary string ansatz (which can be used to solve the equations of motion in any stationary spacetime) by solving the equations of motion in flat spacetime. This will prove instructive when we investigate black hole spacetimes in the next chapter.

In Cartesian coordinates the Minkowski metric can be written as follows

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.32}$$

We will look for solutions of the form

$$x^\mu(\zeta^A) = (\tau, X^i(\sigma)), \tag{2.33}$$

where τ and σ parameterize the world sheet and $X^i(\sigma) = (X(\sigma), Y(\sigma), Z(\sigma))$. Again, we identify the world-sheet coordinate τ with the Minkowski time coordinate t. Thus the spatial configuration of the string is independent of time. We can write down the induced metric G_{AB} on such a world sheet:

$$\begin{aligned} dS^2 &= G_{AB}dX^A dX^B, \\ &= -d\tau^2 + (X'^2 + Y'^2 + Z'^2)d\sigma^2, \end{aligned} \tag{2.34}$$

where the prime denotes differentiation by σ. The determinant of this induced metric is given by

$$-G = X'^2 + Y'^2 + Z'^2. \tag{2.35}$$

Inserting (2.35) into the action (1.20) one can write down the Euler-Lagrange equations:

$$\frac{d}{d\sigma}\left(\frac{X^{i\prime}}{\sqrt{-G}}\right) = 0. \tag{2.36}$$

Integrating these equations we find the solution X^i satisfies

$$\frac{X'}{A} = \frac{B'}{B} = \frac{Z'}{C}. \tag{2.37}$$

Thus the solutions are infinitely long straight lines.

If the string is lying along the x-axis, for example, the spacetime stress-energy tensor of the string (2.21) is found to be

$$T^{\mu\nu}(x) = \mu(-1,1,0,0)\int d\sigma\delta^{(3)}(\mathbf{x} - \mathbf{X}). \tag{2.38}$$

Thus we note that the tension of the string is equal to the mass per unit length. This is a general feature of relativistic strings. However, if one integrates out the small-scale structure of a string [25] by averaging over some length scale one obtains an effective stress-energy for which the tension and the linear mass density are not equal. We will not consider this situation further here.

Chapter 3

Rigidly Rotating Strings in Black Hole Spacetimes

The capture of cosmic strings by black holes has been studied by Lonsdale and Moss [26] and more recently by De Villiers and Frolov [27]. Numerical investigations show that strings encountering a black hole can be captured by the black hole. Numerical investigations of strings moving with a velocity v relative to the black hole find that a black hole of radius $2GM$ has a capture cross-section that scales as some power of $2GM/v$. The cross-section also depends on the size of the string.

The evolution of a string and a black hole system can be very complicated. However, after sufficient time one might expect the interaction to settle down into a stationary state. For this reason we are interested in analytic string solutions in black hole spacetimes that may be useful in studying the behaviour of a trapped string or loops formed during a close encounter. In stationary spacetimes it is often possible to find solutions of the equations of motion by looking for stationary string configurations [14].

In the general case a stationary string in a stationary spacetime is defined as a timelike minimal surface that is tangent to the Killing vector generating time translations. In the Kerr-Newman metric the equations describing a stationary string allow separation of variables [14, 28, 29] and can be solved exactly [14]. In this chapter we generalize these results to a wider class of string configurations. Namely, we study rigidly rotating strings in a stationary axisymmetric background spacetime. A rigidly rotating string is a string which at different moments of time has the same form so that its configuration at later moment of time can be obtained by the rigid rotation of

the initial configuration around the axis of symmetry. We will denote by $\xi_{(t)}$ and $\xi_{(\phi)}$ Killing vectors that are generators of time translation and rotation respectively. The timelike minimal worldsheets which represent a stationary rigidly rotating string are characterized by the property that the linear combination $\xi_{(t)} + \Omega\xi_{(\phi)}$, which is also a Killing vector, is tangent to the worldsheet. Our aim is to study such configurations in a stationary axisymmetric spacetime.

The chapter is organized as follows. General equations for a stationary rigidly rotating string in a stationary spacetime are obtained and analyzed in Section 2.2. As the simplest application we obtain explicit analytical solutions describing rotating strings in a flat spacetime (Section 2.3). One of the interesting results is the possibility of the rigid rotation of the string with (formally) superluminal velocity, i.e. when $r\Omega > 1$ (r is the distance from the axis of rotation). A simple explanation of this phenomenon is given in Section 2.3. Section 2.4 is devoted to rigidly rotating strings in the Kerr spacetime. To conclude Section 2.4 we present a classification of this new family of rigidly rotating test string solutions in the Kerr spacetime. Non-rotating stationary configurations in the Kerr-Newman spacetime will be dealt with in depth in chapter 4. Finally we make some comments on gravitational radiation by trapped cosmic strings.

3.1 General Equations for Stationary Axisymmetric Spacetimes

Consider a stationary axisymmetric spacetime. Such a spacetime possesses at least two commuting Killing vectors: $\xi_{(t)}$ and $\xi_{(\phi)}$. A Killing vector, ξ, is a vector which satisfies the condition

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0. \tag{3.1}$$

Note that the linear combination of the two $A\xi_{(t)}+B\xi_{(\phi)}$ (where A and B are constant) is also a Killing vector.

If the spacetime is asymptotically flat the vector $\xi_{(t)}$ is singled out by the requirement that it is timelike at infinity. The vector $\xi_{(\phi)}$ is spacelike at infinity and it is singled out by the property that its integral curves are closed lines. The metric for a stationary axisymmetric spacetime can be written in the form

$$ds^2 = -V\left[dt - wd\phi\right]^2 + \frac{1}{V}\left[\rho^2 d\phi^2 + e^{2\gamma}(d\rho^2 + dz^2)\right], \tag{3.2}$$

where V, w and γ are functions of the coordinates ρ and z only. This is the Papapetrou form of the metric for stationary axisymmetric spacetimes (see [18] for example). In these coordinates $\xi^{\mu}_{(t)} = \delta^{\mu}_{t}$ and $\xi^{\mu}_{(\phi)} = \delta^{\mu}_{\phi}$.

Let S be a two-dimensional timelike minimal surface representing the motion of a string in this spacetime and denote by S_t the spatial slice t = const. The intersection of S with the surface S_t is a one-dimensional line γ_t representing the string configuration at the time t. We define a rigid cosmic string as one whose shape and extent (but not necessarily position) are independent of the coordinate time t. If x^i are spatial coordinates (for metric (3.2) (ρ, z, ϕ)) then γ_t is given by the equations $x^i = x^i(\sigma, t)$, where σ is a parameter along the string. Since $\xi_{(\phi)}$ is tangent to S_t it is a generator of symmetry transformations (spatial rotations) acting on S_t. It is evident that this transformation preserves the form and the shape of the string γ_t. Our assumption that the string at the moment t is obtained by a rigid rotation from the string γ_{t_0} can be written as

$$\rho(\sigma, t) = \rho(\sigma, t_0), \quad z(\sigma, t) = z(\sigma, t_0), \quad \phi(\sigma, t) = \phi(\sigma, t_0) + p(t, t_0). \tag{3.3}$$

Moreover we assume uniform rotation, so that $p(t, t_0) = \Omega(t - t_0)$, where Ω is a constant angular velocity. Thus the combination $\chi^{\mu} = \xi^{\mu}_{(t)} + \Omega\xi^{\mu}_{(\phi)}$ of the Killing vectors $\xi^{\mu}_{(t)}$ and $\xi^{\mu}_{(\phi)}$ is tangent to the worldsheet S of a uniformly rotating string.

In a region where χ^μ is timelike one can define a set of Killing observers whose four-velocities are $u^\mu = \chi^\mu / |\chi^2|^{1/2}$. This set of observers form a rigidly rotating reference frame that is the frame moving with a constant angular velocity Ω. One could choose to define a rigidly rotating string as a string which was fixed in form and position in the frame of some Killing observer with angular velocity Ω. It can be shown that if all the string is located in the region where χ^μ is timelike this definition is equivalent to that given above. But, as we shall demonstrate later, a rigidly rotating string can lie in a region where the Killing vector χ^μ is spacelike, while its world sheet surface S remains timelike. With this possibility in mind we will use the former definition of the rigid string rotation.

We begin by performing the following coordinate transformation:

$$\varphi = \phi - \Omega t, \tag{3.4}$$

where Ω is a constant. The metric (3.2) now takes the form

$$ds^2 = -V[(1-\Omega w)dt - wd\varphi]^2 + \frac{1}{V}\left[\rho^2(d\varphi + \Omega dt)^2 + e^{2\gamma}(d\rho^2 + dz^2)\right] \tag{3.5}$$

and the Killing vector χ has components $\chi^\mu = (1,0,0,0)$. The Killing trajectories of χ (that might be timelike or spacelike) are: ρ, z, φ =const .

Let the coordinates on the world sheet S be $(\zeta^0, \zeta^1) = (\tau, \sigma)$. For a stationary world sheet configuration one can choose parameters (τ, σ) in such a way that

$$x^\mu(\zeta^A) = (\tau + f(\sigma), \rho(\sigma), z(\sigma), \varphi(\sigma)), \tag{3.6}$$

where f is some function of σ. The determinant of the induced metric G_{AB} on the world sheet is

$$G = \frac{e^{2\gamma}\chi^2}{V}\left(\rho'^2 + z'^2\right) - \rho^2\varphi'^2. \tag{3.7}$$

where

$$\chi^2 = -V + 2\Omega w V + \Omega^2(\rho^2/V - w^2 V), \tag{3.8}$$

Note that neither the function f nor its derivative f' appear in the action so they may be specified freely. It is convenient to choose f so that the induced metric is diagonal, i.e. $G_{\tau\sigma} = g_{\mu\nu}x^{\mu}_{,\tau}x^{\nu}_{,\sigma} = 0$. In this case we find f must be chosen to satisfy the condition

$$f' = -\frac{Vw + \Omega(\rho^2/V - w^2V)}{\chi^2}\,\varphi'. \tag{3.9}$$

A stationary string configuration (3.6) provides an extremum for the reduced Nambu-Goto action

$$E = \mu \int d\sigma \sqrt{\frac{e^{2\gamma}(-\chi^2)}{V}\left(\rho'^2 + z'^2\right) + \rho^2\varphi'^2}. \tag{3.10}$$

Hence a stationary string configuration $x^i = (\rho(\sigma), z(\sigma), \varphi(\sigma))$ is a geodesic line in a three-dimensional space with the metric

$$dh^2 = \frac{e^{2\gamma}(-\chi^2)}{V}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\,. \tag{3.11}$$

Frolov *et al* [14] exploit this fact to solve the equations of motion for rigid non-rotating string ($\Omega = 0$) in the Kerr-Newman spacetime. We will study these solutions in section 3.3.2 using an alternative approach.

The Nambu-Goto equations for a stationary rigidly rotating string are

$$\partial_\sigma\left(\frac{\chi^2 e^{2\gamma}}{\sqrt{-G}V}\rho'\right) = \frac{1}{\sqrt{-G}}\left[\frac{1}{2}\frac{\partial}{\partial\rho}\left(\frac{\chi^2 e^{2\gamma}}{V}\right)\left(\rho'^2 + z'^2\right) - \rho\varphi'^2\right], \tag{3.12}$$

$$\partial_\sigma\left(\frac{\rho^2\varphi'}{\sqrt{-G}}\right) = 0, \tag{3.13}$$

$$\partial_\sigma\left(\frac{\chi^2 e^{2\gamma}}{\sqrt{-G}V}z'\right) = \frac{1}{2\sqrt{-G}}\frac{\partial}{\partial z}\left(\frac{\chi^2 e^{2\gamma}}{V}\right)\left(\rho'^2 + z'^2\right), \tag{3.14}$$

where G is given by (3.7). Equation (3.13) can be integrated immediately to give

$$\varphi'^2 = \frac{-\chi^2}{V}\left(\frac{L^2 e^{2\gamma}}{\rho^2(\rho^2 - L^2)}\right)(\rho'^2 + z'^2). \tag{3.15}$$

Here L is a constant of the integration. The constant L is associated with the φ-independence of the Lagrangian and is related to the angular momentum of the string. In what follows we choose L to be non-negative.

These equations are invariant under the reparameterization $\sigma \to \tilde{\sigma} = \tilde{\sigma}(\sigma)$. In the region where $\rho' \neq 0$ one can use this ambiguity to put $\sigma = \rho$. With this choice equations (3.12)-(3.14) reduce to

$$\left(\frac{d\varphi}{d\rho}\right)^2 = \frac{-\chi^2 L^2 e^{2\gamma}}{\rho^2(\rho^2 - L^2)V}\left[1 + \left(\frac{dz}{d\rho}\right)^2\right], \tag{3.16}$$

$$\frac{d}{d\rho}\left(\frac{\chi^2 e^{2\gamma}}{\sqrt{-\tilde{G}}V}\frac{dz}{d\rho}\right) = \frac{1}{2\sqrt{-\tilde{G}}}\frac{\partial}{\partial z}\left(\frac{\chi^2 e^{2\gamma}}{V}\right)\left[1 + \left(\frac{dz}{d\rho}\right)^2\right], \tag{3.17}$$

where

$$-\tilde{G} = \frac{-G}{\rho'^2} = \frac{\rho^2 \chi^2 e^{2\gamma}}{V(L^2 - \rho^2)}\left[1 + \left(\frac{dz}{d\rho}\right)^2\right]. \tag{3.18}$$

The solutions represent a timelike two-surface provided the determinant G is negative definite. Thus we see that the rigidly rotating strings are confined to regions (for $V > 0$) where

$$I \equiv \frac{L^2 - \rho^2}{\chi^2} > 0. \tag{3.19}$$

When $L^2 = \rho^2$ the world sheet has a turning point in ρ as a function of φ.

In general, in order to ensure rigid rotation of a string, an external force must act on it. For example, one could assume that a string has heavy monopoles at the ends and that a magnetic field is applied to force them to move along a circle. In this case a solution of equations (3.16)-(3.17) describes the motion of the string interior. In order to escape a discussion of the details of the motion of the end points we shall use the maximal extensions of the string solutions, continuing them until they meet the surface where $\chi^2 = 0$. Since the invariant I changes its sign at this surface, the minimal surface describing the rigidly rotating string ceases to be timelike here. The end points of such a maximally extended string move with the velocity of light along this surface (equivalently the solutions satisfy Neumann boundary conditions). We call such solutions "open" strings.

There are instances when the rigidly rotating string ansatz does give solutions which are not "open" in the above sense. In Section 3.3 we will see there are solutions in the Kerr-Newman spacetime which have end-points on the black hole. Formally this is equivalent to the "open" string boundary conditions since the horizon is a null surface. Also if $\Omega = 0$, so that our solutions, are not rotating the maximally extended solutions have end-points at spatial infinity. Again, one could have monopoles on the ends of the string, far from the black hole, to which some force is applied to prevent the string falling into the black hole.

Our assumption of rigidity implies that the coordinates ρ and z of the end points of the string remain fixed. Under these conditions the end points of an "open" string are located on the surfaces where $\chi^2 = 0$. In a flat spacetime this timelike surface is a cylinder located at the radial distance Ω^{-1} from the axis of symmetry. In the general case we shall refer to the surfaces where $\chi^2 = 0$ as "null cylinders". Note that if $L^2 - \rho^2$ vanishes at the same point as χ^2 it is possible for the world sheet to pass through the surface $\chi^2 = 0$ and remain regular and timelike.

In order to find a rigidly rotating string configuration one needs to fix functions $V(\rho, z)$, $\gamma(\rho, z)$, and $w(\rho, z)$ that specify geometry. It is quite interesting to note that (as was remarked by De Vega and Egusquiza [30]) that if the metric (3.2) allows a discrete symmetry $z \to -z$, then equations (3.16) and (3.17) always have a special solution, namely a string configuration described by the relations $z = 0$ and $\varphi =$ const. De Vega and Egusquiza called these straight rigidly rotating strings in axially-symmetric stationary spacetimes "planetoid" solutions.

3.2 Rotating Strings in Flat Spacetime

Our main goal is a study of rigidly rotating strings in the spacetime of a rotating black hole. But before considering this problem we make a few remarks concerning rigidly rotating strings in a flat spacetime. We recover the Minkowski metric

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2. \tag{3.20}$$

in cylindrical coordinates from (3.2) by setting the metric functions $V = 1$ and $w = \gamma = 0$. We also have $\chi^2 = \Omega^2\rho^2 - 1$. Since the metric is independent of z one can integrate (3.17) once to reduce the equations of motion to the form

$$\frac{d\varphi}{d\rho} = \pm\frac{L\,(1-\Omega^2\rho^2)}{\rho\sqrt{(1-p^2+L^2\Omega^2)\,\rho^2-\Omega^2\rho^4-L^2}}, \tag{3.21}$$

$$\frac{dz}{d\rho} = \pm\frac{p\,\rho}{\sqrt{(1-p^2+L^2\Omega^2)\,\rho^2-\Omega^2\rho^4-L^2}}, \tag{3.22}$$

where p is a constant of integration. These equations can be solved analytically.

In order for (3.21) and (3.22) to be real-valued, ρ is constrained to lie in the interval $0 < \rho_- \leq \rho < \rho_+$ where the upper and lower bounds are given by,

$$\rho_\pm = \Omega^{-1}\sqrt{(B \pm C)/2}\,, \tag{3.23}$$

where,

$$B = 1 - p^2 + L^2\Omega^2, \text{ and } C = \sqrt{B^2 - 4\Omega^2 L^2}\,. \tag{3.24}$$

The equations for $z(\rho)$ and $\varphi(\rho)$ can be integrated readily (the substitution $u = \rho^2$ reduces these to standard integrals) with solutions,

$$z_\pm(\rho) = \mp\frac{p}{2\Omega}\left(\arcsin\frac{B-2\Omega^2\rho^2}{C} - \frac{\pi}{2}\right), \tag{3.25}$$

$$\varphi_\pm(\rho) = \pm\frac{1}{2}\left\{\arcsin\frac{B\rho^2-2L^2}{C\rho^2} + \Omega L\arcsin\frac{B-2L^2\rho^2}{C} + \frac{\pi}{2}(1-\Omega L)\right\} \tag{3.26}$$

where for convenience we have chosen the initial conditions $z_\pm(\rho_-) = \varphi_\pm(\rho_-) = 0$.

It is instructive to examine the special case where $p = 0$ further where the solutions are confined to the z = const plane. The solution (3.26) can be rewritten in the form

$$\varphi_\pm(\rho) = \pm \left(\arctan k\,\xi - k^{-1} \arctan \xi \right) , \tag{3.27}$$

where $k = (\Omega L)^{-1}$ and $\xi = \Omega\sqrt{(\rho^2 - L^2)/(1 - \Omega^2\rho^2)}$.

In order for solutions to exist, the invariant $I \equiv (L^2 - \rho^2)/\chi^2$ must be non-negative. Thus there are a number of cases to resolve. We know that the string can end only on the null cylinder where χ^2 vanishes, i.e. $\rho = 1/\Omega$. We also see that the string may have a turning point at $\rho = L$. When $L = 0$ we recover the rigidly rotating straight strings of De Vega and Egusquiza [30]. When $L > 0$ there are two cases;

1. $L < 1/\Omega$: the string lies in the region $L < \rho < 1/\Omega$, has end-points at $\rho = 1/\Omega$ and a turning point at $\rho = L$ (see figure 3.2.1),

2. $L > 1/\Omega$: the string lies in the region $L > \rho > 1/\Omega$. It has end-points at $\rho = 1/\Omega$ and a turning point at $\rho = L$ (see figure 3.2.2).

(The case $L = 1/\Omega$ is excluded since $I < 0$ and hence no solution exists.)

In the latter case the Killing vector χ is spacelike. Nonetheless the world-sheet is timelike; in fact, the tangent vector $x^\mu_{,\sigma}$ is timelike in this region. However, the solution lies in the region $L > \rho > 1/\Omega$ and appears, by comparing t = const slices in non-rotating coordinates, to move at "superluminal velocities" (except at the end points which move at the speed of light). This is in apparent contradiction with the observation that the world sheet is timelike.

The puzzle is clarified if we note that the apparent velocity of the string in the surface t = const is not the physical velocity of the string. Recall that the

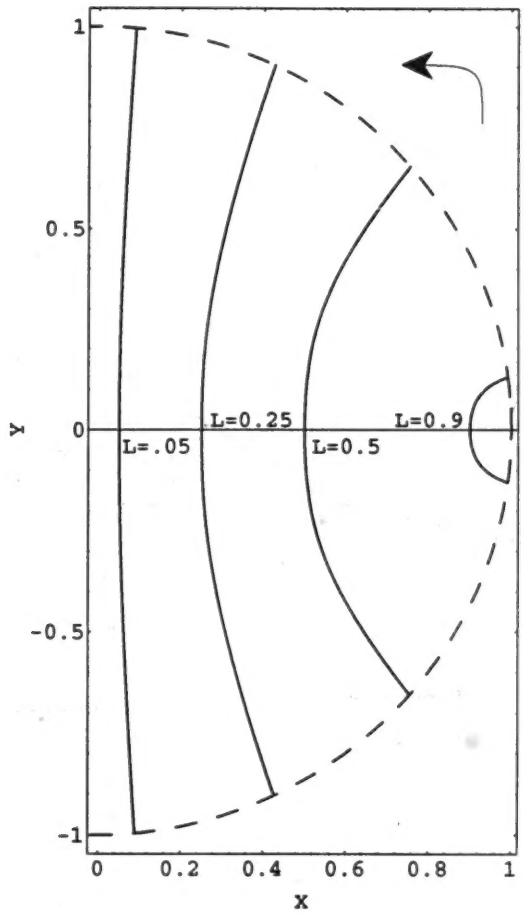

Figure 3.2.1: String configurations in flat spacetime for $p = 0$ with $L < 1/\Omega$. Solid lines represent strings for 4 different values $L = 0.05, 0.25, 0.5$, and 0.9 of the angular momentum. A dashed line is a null cylinder $\rho = 1/\Omega$ (here $\Omega = 1$). The arrow in this and subsequent figures indicates the direction of rotation of the strings.

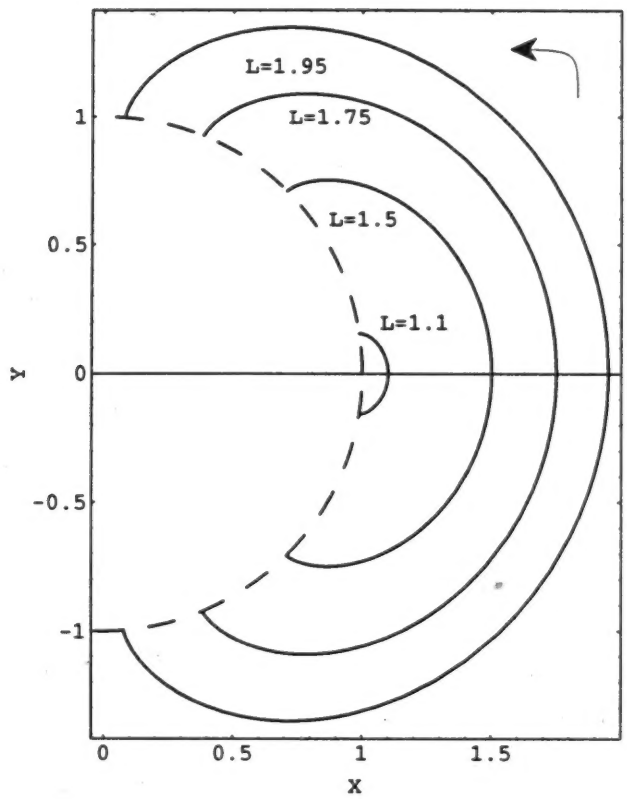

Figure 3.2.2: String configurations in flat spacetime for $p = 0$ with $L > 1/\Omega$. Solid lines represent strings for 4 different values $L = 1.1, 1.5, 1.75$, and 1.95 of the angular momentum. A dashed line is a null cylinder $\rho = 1/\Omega$ (here $\Omega = 1$).

Nambu-Goto action is invariant under world-sheet reparameterizations. This reparameterization can be used to generate a "motion" of the string along itself, which evidently is physically irrelevant. In other words, only the velocity component normal to the string world-sheet has physical meaning (see [1]).

Hence, on the $t = \text{const}$ hypersurfaces, we must consider the component of the apparent string velocity normal to the string configuration. The normal component of the velocity is the physical component. The apparent three velocity, v^i $(i = 1, 2, 3)$, of the string in (ρ, z, ϕ) coordinates as measured by a static observer at infinity is

$$v^i = \Omega(0, 0, 1) \tag{3.28}$$

and has magnitude $v = \rho\Omega$.

Its projection on the normal to the string $u^i_\perp$ in the $(t = \text{const})$ plane is

$$u^i_\perp = \frac{\Omega}{1 + \rho^2\varphi'^2}(-\rho^2\varphi', 0, 1), \tag{3.29}$$

and the magnitude of this normal velocity is

$$u^2_\perp = \frac{\rho^2\Omega^2}{1 + \rho^2\varphi'^2} = \frac{\Omega^2(L^2 - \rho^2)}{\Omega^2 L^2 - 1}. \tag{3.30}$$

Thus we see that if $1/\Omega^2 > \rho^2 > L^2$ (case 1) then $u^2_\perp < 1$ as expected. Furthermore if $L^2 > \rho^2 > 1/\Omega^2$ (the apparently "superluminal" case 2) we see that $u^2_\perp < 1$ also. The physical velocity of the string is subluminal in all cases where the solution exists.

This phenomenon is evidently of a quite general nature. In order to separate these two different types of rigid rotation of strings we will call the motion "superluminal" if they are tangent to χ with $\chi^2 > 0$ and "subluminal" if the world-sheet is tangent to χ with $\chi^2 < 0$.

3.3 Rigidly Rotating Strings in the Kerr-Newman Spacetime

The Kerr-Newman spacetime is that of an electrically charged, rotating black hole (see [31] for example). In Boyer-Lindquist coordinates [32] the Kerr-Newman metric is given by:

$$\begin{aligned} ds^2 = & - \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta d\phi\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ & + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2, \end{aligned} \tag{3.31}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. These metrics form a three parameter family with parameters M, a and Q. When $Q = 0$ we have the Kerr family of solutions. When $a = 0$ we recover the Reissner-Nordstrom family of solutions. Finally when $a = Q = 0$ the metric (3.31) reduces to the Schwarzchild solution. All asymptotically flat stationary black hole solutions of the Einstein-Maxwell equations are encompassed by this family.

This spacetime is stationary and axisymmetric with two Killing vectors given by $\xi^\mu_{(t)} = \delta^\mu_t$ and $\xi^\mu_{(\phi)} = \delta^\mu_\phi$ in the Boyer-Lindquist coordinates. The norm of the Killing vector $\xi^\mu_{(t)}$ is:

$$F \equiv -\xi^2 = 1 - \frac{2Mr - Q^2}{\rho^2}. \tag{3.32}$$

A surface S_{st} where ξ becomes null ($F = 0$) is known as the static limit surface. It is defined by:

$$r = r_{st} \equiv M + \sqrt{M^2 - Q^2 - a^2\cos^2\theta}. \tag{3.33}$$

We now consider the equations of motion for rigidly rotating strings (3.14-3.16) in the Kerr-Newman spacetime.

3.3.1 Equations of Motion

The relationship between the Boyer-Linquist coordinate functions of the Kerr-Newman metric and the Papapetrou coordinate functions is straightforward; the time coordinate t and the angular coordinate ϕ that appear in both the metric (3.2) and the metric (3.31) are simply identified and

$$\rho^2 = \Delta \sin^2\theta\,, \qquad z = (r-M)\cos\theta. \tag{3.34}$$

In terms of the Boyer-Lindquist coordinates the Papapetrou metric functions for the Kerr-Newman metric are

$$V = \frac{\Delta - a^2\sin^2\theta}{\Sigma}, \tag{3.35}$$

$$w = \frac{(\Delta - (r^2+a^2))a\sin^2\theta}{\Delta - a^2\sin^2\theta}, \tag{3.36}$$

$$e^{2\gamma} = \frac{\Delta - a^2\sin^2\theta}{\Delta\cos^2\theta + (r-M)^2\sin^2\theta}. \tag{3.37}$$

We can now write down the string equations of motion for the Kerr-Newman spacetime in the Boyer-Linquist coordinates. The string configuration is determined by two functions $\varphi(r)$ and $\theta(r)$ which satisfy

$$\left(\frac{d\varphi}{dr}\right)^2 = \frac{\tilde{G}L^2}{\Delta^2\sin^4\theta} \tag{3.38}$$

$$\frac{1}{\sqrt{-G}}\frac{d}{dr}\left(\frac{\Sigma\chi^2}{\sqrt{-G}}\frac{d\theta}{dr}\right) = -\frac{\cos\theta}{\sin^3\theta}Z\,, \tag{3.39}$$

where

$$Z = \sin^2\theta - b^2\left(\frac{1-a^2\Omega^2\sin^4\theta}{\Sigma\chi^2}\right) \tag{3.40}$$

$$b^2 = L^2 - \Delta\sin^2\theta\,, \quad -G = \frac{\Sigma\chi^2\sin^2\theta}{b^2}\left[1+\Delta\left(\frac{d\theta}{dr}\right)^2\right]. \tag{3.41}$$

$$\chi^2 = \frac{\sin^2\theta\left[a^2 + 2\Omega(2Mr+Q^2)a + \Omega^2((r^2+a^2)^2 - \Delta a^2\sin^2\theta)\right] - \Delta}{\Sigma} \tag{3.42}$$

We were not able to find the general solution to this system analytically. However, there are several cases where one can make further progress. Solutions can be found in the non-rotating case where $\Omega = 0$, and in the equatorial plane $\theta = \pi/2$ for any Ω. Note that when $L = 0$, one recovers the planetoid string solutions of De Vega and Egusquiza [30]. We will discuss the non-rotating solutions first.

3.3.2 Non-rotating Stationary Strings

These solutions were first discussed by Frolov *et al* [14]. They were able to separate the equations of motion (3.38-3.39) when $\Omega = 0$, using the Hamilton-Jacobi method. It was shown that the general non-rotating stationary string solution in the Kerr-Newman spacetime can be written:

$$\begin{aligned} \left(\frac{d\phi}{dr}\right)^2 &= \frac{L^2}{\Delta^2 - q^2\Delta + a^2L^2}\left(\frac{\Delta - a^2\sin^2\theta}{\Delta\sin^2\theta}\right)^2, \\ \left(\frac{d\theta}{dr}\right)^2 &= \frac{q^2 - \frac{L^2}{\sin^2\theta} - a^2\sin^2\theta}{\Delta^2 - q^2\Delta + a^2L^2}, \end{aligned} \tag{3.43}$$

where q is an arbitrary constant. The solutions described here are all infinite string solutions; these boundary conditions model strings whose end-points are very far from the black hole. In particular, Frolov *et al* [14] analyse the equations (3.43) when the parameters satisfy the relation

$$q^2 = \frac{L^2}{\sin^2\theta_0} + a^2\sin^2\theta_0, \tag{3.44}$$

where θ_0 is the value that minimises q^2. In this case the solution of (3.43) lies on the cone $\theta = \theta_0$. There are two subcases to distinguish:

1. if $L^2 \geq a^2$ then $q^2 = L^2 + a^2$ and $\theta_0 = \pi/2$ i.e. the string lies in the equatorial plane. The string configuration $\phi = \phi(r)$ is given by

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{L^2}{\Delta^2}\left(\frac{\Delta - a^2}{\Delta - L^2}\right)^2. \tag{3.45}$$

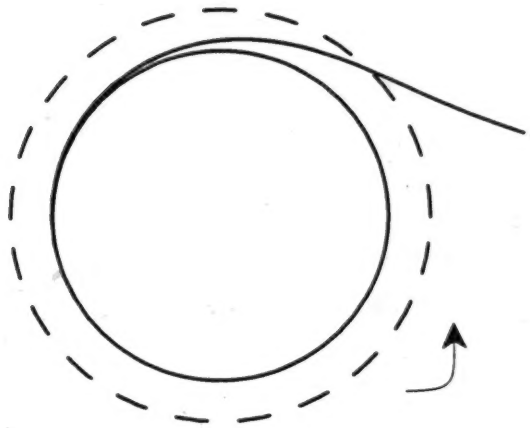

Figure 3.3.1: An equatorial cone string configuration (3.46) is shown near the black hole. The dashed line is the static limit and the solid line is the horizon of the black hole. The string passes through the static limit and spirals into the horizon.

Note that the string has a turning point at $\Delta = L^2$ outside the static limit unless $L^2 = a^2$. If $L^2 = a^2$ then the string passes through the static limit and spirals into the horizon. Figure 3.3.1 shows such a string configuration ($\phi' = a/\Delta$) in a region near the black hole.

2. if $L^2 \leq a^2$ then $q^2 = 2a|L|$ and $\sin^2\theta_0 = |L|/a$. These solutions lie on the cone $\theta = \theta_0$; the configuration $\phi = \phi(r)$ is given by:

$$\left(\frac{d\phi}{dr}\right)^2 = \frac{a^2}{\Delta^2}. \tag{3.46}$$

These configurations all describe a string which passes through the static limit and spirals into the horizon of the black hole. These configurations are called the *cone strings* and will be studied in detail in chapter 4.

3.3.3 Rigidly Rotating Strings in the Equatorial Plane

We now consider the rigidly rotating solutions ($\Omega \neq 0$) in the equatorial plane. In what follows we consider the case when a rotating string is located in the equatorial plane of a Kerr black hole ($Q = 0$ and $\theta = \pi/2$); the assumption that $Q = 0$ is not necessary here but simplifies our considerations without altering the analysis substantially.

For the motion of the string in the equatorial plane $\theta = \pi/2$ of the Kerr spacetime ($Q = 0$) equation (3.39) is satisfied identically (both the left and right hand sides vanish) and equation (3.38) takes the form

$$\left(\frac{d\varphi}{dr}\right)^2 = \frac{\Sigma L^2 \chi^2}{\Delta^2 (L^2 - \Delta)}. \tag{3.47}$$

Here

$$\chi^2 = \frac{H}{r}, \quad H = \Omega^2 r^3 + (a^2\Omega^2 - 1)r + 2M(a\Omega - 1)^2. \tag{3.48}$$

Solutions exist only if the right-hand side of (3.47) is non-negative and hence (for $L \neq 0$)

$$I \equiv \frac{L^2 - \Delta}{\chi^2} \geq 0. \tag{3.49}$$

The positivity of the invariant I also guarantees that the world-sheet of the string is a regular timelike surface (c.f. (3.19)).

To simplify the analysis of the structure of the null cylinder surfaces instead of r, a, L, Ω and H we introduce the dimensionless variables:

$$\rho = \frac{r}{M}, \quad \alpha = \frac{a}{M}, \quad \lambda = \frac{L}{M}, \quad \omega = M\Omega, \quad H = Mh. \tag{3.50}$$

In these variables

$$\chi^2 = \frac{h}{\rho}, \quad h = \omega^2 \rho^3 + (\alpha^2\omega^2 - 1)\rho + 2(\alpha\omega - 1)^2. \tag{3.51}$$

The surface where $L^2-\Delta$ vanishes corresponds, in general, to turning points of the string in r. Now $L^2-\Delta$ has only one zero outside the horizon, namely $r_0 = M\rho_0$ with

$$\rho_0 = 1+\sqrt{1+\lambda^2-\alpha^2}\,. \tag{3.52}$$

For $r > r_0$, $L^2-\Delta < 0$, and for $r < r_0$, $L^2-\Delta > 0$.

First we note that when $L = 0$ then we obtain the planetoid solution of De Vega and Egusquiza [30]. In this case the solution is a rigidly rotating straight string with end points on the null cylinders $r = r_1$ and $r = r_2$ where $r_2 > r_1$ are the zeroes of χ^2. For a given Ω, if these zeroes do not exist (so that χ is spacelike everywhere) then there are no such rigidly rotating straight strings.

In the general case the endpoints of an "open" string must be located on the null cylinders where $\chi^2 = 0$. Note one can think of the equation $h(\rho,\omega,\alpha) = 0$ as a quadratic in ω for fixed ρ and α. The zeroes of (3.51), that is the solutions of the equation $h(\rho,\omega,\alpha) = 0$, are

$$\omega_\pm = \frac{2\alpha \pm \rho\sqrt{\rho^2-2\rho+\alpha^2}}{\rho^3+\alpha^2\rho+2\alpha^2}. \tag{3.53}$$

They bound the interval $\omega_-(\rho) < \omega < \omega_+(\rho)$ where χ is timelike at a given radius $r = M\rho$. We note that inside the horizon where $\rho^2-2\rho+\alpha^2 < 0$, it is not possible for χ to be timelike or null except within, or on, the inner Cauchy horizon. Since we are interested in the motion of the strings in the black hole exterior from now on we restrict ourselves to solutions in the region $\rho > \rho_+$ where

$$\rho_+ = 1+\sqrt{1-\alpha^2}. \tag{3.54}$$

Equation (3.53) shows that $\omega_\pm|_{r_+} = \omega_{BH} \equiv \alpha/(\rho_+^2+\alpha^2)$ (ω_{BH}/M is the angular velocity of the black hole). At large distances $\omega_\pm = \pm 1/\rho$ reproduces flat space behavior. For fixed α the function $\omega_+(\rho,\alpha)$ has a maximum, and the function

$\omega_-(\rho,\alpha)$ has a minimum. The points of the extrema can be defined as joint solutions of the equation $h(\rho,\omega,\alpha)=0$ and the equation $\partial h/\partial\rho|_{(\omega,\alpha)}=0$. The latter equation implies that

$$\rho=\pm\rho_m\,,\quad \rho_m=\frac{\sqrt{1-\omega^2\alpha^2}}{\sqrt{3}|\omega|}\,. \tag{3.55}$$

A simultaneous solution of this relation and equation (3.53) defines a maximal value $\omega_{max}(\alpha)$ of ω_+ and a minimal value $\omega_{min}(\alpha)$ of ω_-. We conclude that for a given value of the rotation parameter α the Killing vector χ can be timelike only if $\omega_{min}(\alpha)<\omega<\omega_{max}(\alpha)$. If there exists a region where the Killing vector χ is timelike, this region is inside interval $\rho_1<\rho<\rho_2$, and one has $\rho_{min}(\alpha)<\rho_1$ and $\rho_{max}(\alpha)>\rho_2$ (where $\rho_1(\omega,\alpha)<\rho_2(\omega,\alpha)$ are the zeroes of the polynomial h). Here $\rho_{min}(\alpha)$ and $\rho_{max}(\alpha)$ are given by (3.55) with $\omega=\omega_{min}(\alpha)$ and $\omega=\omega_{max}(\alpha)$, respectively.

We can arrive at the same conclusion by slightly different reasoning which will allow us to make further simplifications. For fixed α and ω the function $h(\rho,\omega,\alpha)$ is a cubic polynomial in ρ . It tends to $\pm\infty$ as $\rho\to\pm\infty$ and $h(0)\geq 0$ ($h(0)=0$ only if $\omega\alpha=1$). For $\alpha|\omega|\geq 1$ it is monotonic, and hence always positive at $\rho>0$. For $\alpha|\omega|<1$ the function $h(\rho)$ has a minimum at $\rho=\rho_m$ and a maximum at $\rho=-\rho_m$, where ρ_m is given by (3.55). At the minimum point, h takes the value

$$h_m=h(\rho_m,\omega)=2(\omega\alpha-1)^2\left[1-\frac{1}{3\sqrt{3}|\omega|}\sqrt{\frac{(1+\omega\alpha)^3}{1-\omega\alpha}}\right]. \tag{3.56}$$

The minimum value h_m vanishes if the following equation is satisfied

$$\alpha(\alpha^2+27)\omega^3+(3\alpha^2-27)\omega^2+3\alpha\omega+1=0\,. \tag{3.57}$$

Solutions of this equations $\omega(\alpha)$ are also solutions of the two equations $h=0$ and $\partial_r h=0$, and hence they coincide with $\omega_{min}(\alpha)$ and $\omega_{max}(\alpha)$.

The numerical solution of equation (3.57) is shown in figure 3.3.2. Line a represents solution ω_{max} and line b represents solution ω_{min}. These lines begin at

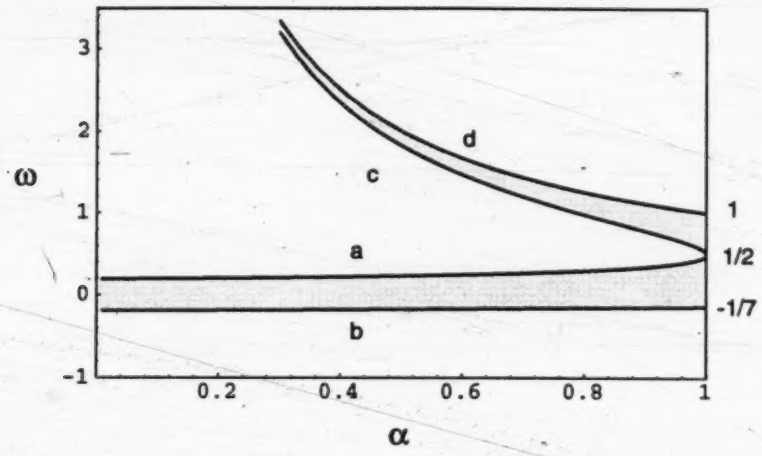

Figure 3.3.2: The roots of equation (3.57) are plotted in the $\alpha - \omega$ plane (curves a, b and c) along with the curve $\omega = 1/\alpha$ (curve d). The shaded regions correspond to parameter values where h has two positive roots; in the upper region between c and d these roots lie inside the inner Cauchy horizon of the black hole and in the lower region between a and b they lie outside the event horizon of the black hole.

$\alpha = 0$ at their Schwarzschild values $\pm 3^{-3/2}$ and reach values $1/2$ and $-1/7$ respectively for the extremely rotating black hole. Figure 3.3.3 shows the corresponding radii ρ_{max} (curve a) and ρ_{min} (curve b) as the functions of the rotation parameter α.

The third branch c in figure 3.3.2, which intersects the curve a at $\alpha = 1$, corresponds to the minimum values of ω inside the Cauchy horizon. Line d is the solution of the equation $\omega\alpha = 1$. For the values of the parameters in the $(\alpha - \omega)$ plane lying in the region outside two shaded strips function h is positive for any $\rho > 0$. For the values of the parameters inside two shaded strips function h has two roots, $0 < \rho_1 < \rho_2$, and h is negative for ρ lying between the roots. The upper shaded region of figure 3.3.2 corresponds to the situation where the roots of h lie within the inner Cauchy horizon. The lower shaded region is the area where the two roots of h lie outside the event horizon.

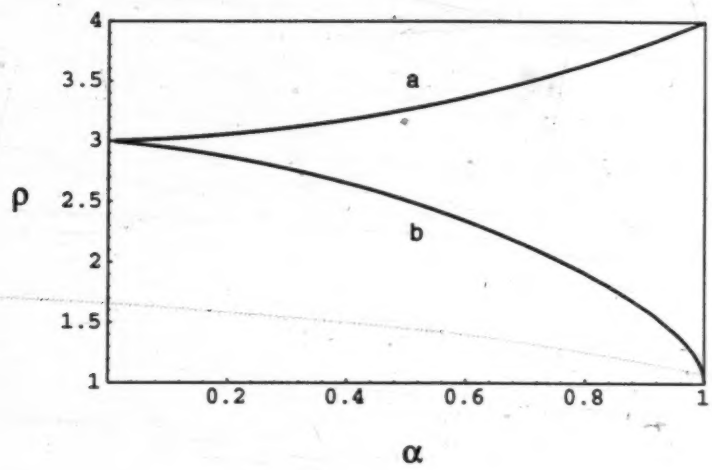

Figure 3.3.3: The functions $\rho_{max}(\alpha)$ (curve a) and $\rho_{min}(\alpha)$ (curve b) are plotted for $0 < \alpha < 1$.

Having obtained this information on the structure of the null cylinder surfaces we now discuss the different types of motion of a rigidly rotating string in the equatorial plane of the Kerr spacetime. The simplest situation clearly occurs when χ has no zeroes in the region $\rho \geq \rho_+$ (where ρ_+ is defined by equation (3.54)). This happens for values of the parameters which lie outside the shaded region restricted by lines a and b in the $(\alpha - \omega)$-plane (see figure 3.3.2). In this case there is only one allowed type of solution $\rho_0 > \rho_+$ corresponding to solutions in the region $\rho_+ \leq \rho \leq \rho_0$. These configurations begin and end in the black hole and have a turning point at $\rho = \rho_0$ in its exterior (the form of these solutions is qualitatively similar to that of the solution shown in figure 3.3.5). Such a solution may describe a closed loop-like string, part of which has been swallowed by a black hole. Centrifugal forces, connected with the rotation of the string, allow it to remain partially outside the horizon.

For the values of the parameters lying inside the shaded region restricted by lines a and b in the $(\alpha - \omega)$-plane the situation is somewhat more complicated. In this

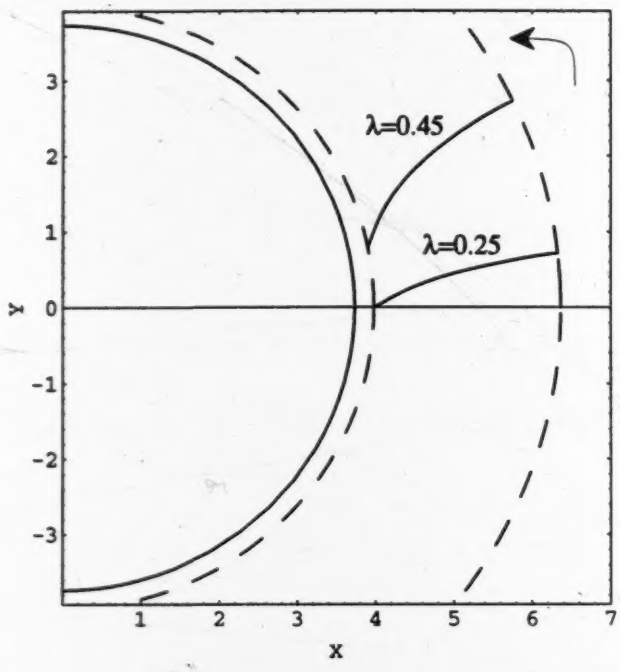

Figure 3.3.4: We show here a typical pair of string configurations ($\lambda = 0.25$, $\lambda = 0.45$) in the region $\rho_1 < \rho < \rho_2$ with no turning points (Case 1(a)). The strings have end-points at $\rho = \rho_1$ and $\rho = \rho_2$ ($\alpha = 0.5, \omega = 0.05$).

case χ^2 has two zeroes that we denote by ρ_1 and ρ_2 (we only consider the case where $\rho_+ < \rho_1 < \rho_2$). The Killing vector χ is timelike in the region $\rho_1 < \rho < \rho_2$. Since the invariant I defined by (3.49) must be positive there are then a number of different possibilities depending upon the choice of the angular momentum parameter $\lambda \geq 0$.

1. $\rho_0 < \rho_1$. The invariant I is positive either if (a) $\rho_1 < \rho < \rho_2$ or (b) $\rho < \rho_0$. In the former case $f < 0$ and the motion of the string is "subluminal", with the ends of the string at ρ_1 and ρ_2 (figure 3.3.4). In the latter case the motion of the string is "superluminal", the string begins and ends on the black hole and has a radial turning point ρ_0 in the black hole exterior (figure 3.3.5).

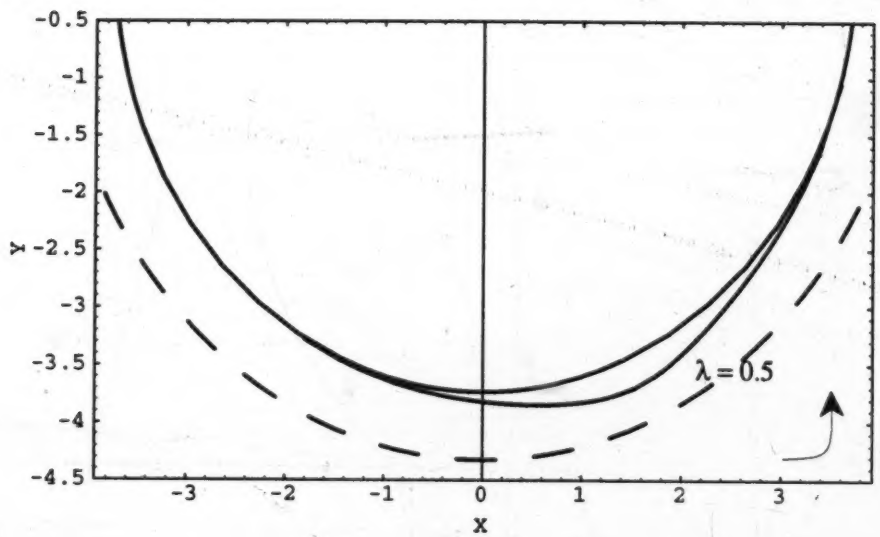

Figure 3.3.5: A string configuration in the region $\rho < \rho_0$ (Case 1(b)). The string has a turning point at $\rho = \rho_0$ ($\alpha = 0.5$, $\omega = 0.06$).

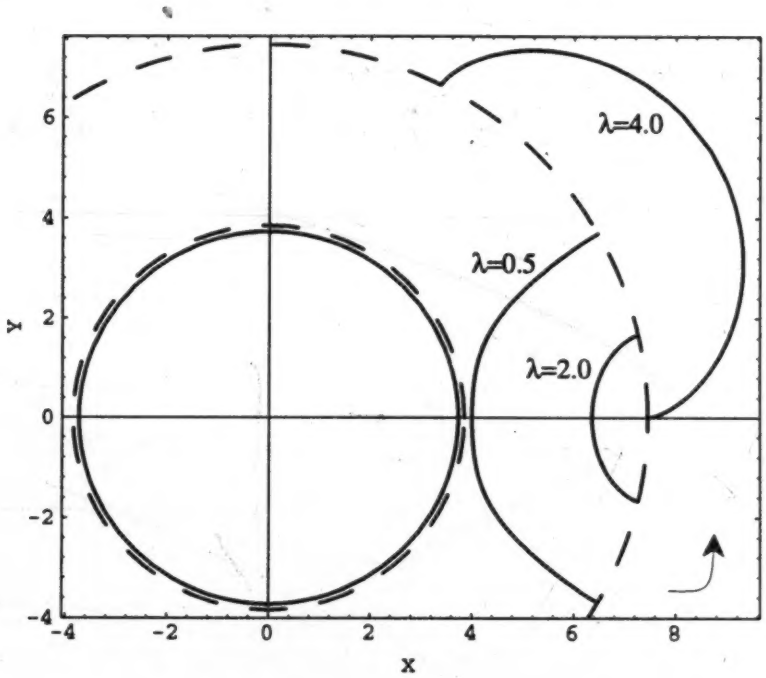

Figure 3.3.6: String configurations inside the region $\rho_0 \leq \rho < \rho_2$ ($\lambda = 0.5$, $\lambda = 2.0$, Case 2(a)) and a typical string configuration in the region $\rho_2 < \rho \leq \rho_0$ ($\lambda = 4.0$, Case 3(b)). All configurations have turning points at ρ_0 ($\alpha = 0.5$, $\omega = 0.05$).

2. $\rho_1 < \rho_0 < \rho_2$. The invariant I is positive either if (a) $\rho_0 < \rho < \rho_2$ or (b) $\rho < \rho_1$. In the former case $h < 0$ and the motion of the string is "subluminal", with both ends of the string at ρ_2 and ρ_0 being radial turning points (two examples (with $\lambda = 0.5$ and $\lambda = 2.0$) are shown in figure 3.3.6). In the latter case the motion of the string is "superluminal" and string begins at the horizon ρ_+ and ends at ρ_1 (figure 3.3.7).

3. $\rho_2 < \rho_0$. The invariant I is positive either if (a) $\rho_2 < \rho < \rho_0$ or (b) $\rho < \rho_1$. In both cases $h > 0$ and the motion is "superluminal". In the former case the ends of the string are at ρ_2 and the radial turning point is at ρ_0 (an example of such a configuration with $\lambda = 4.0$ is given in figure 3.3.6). In the latter case

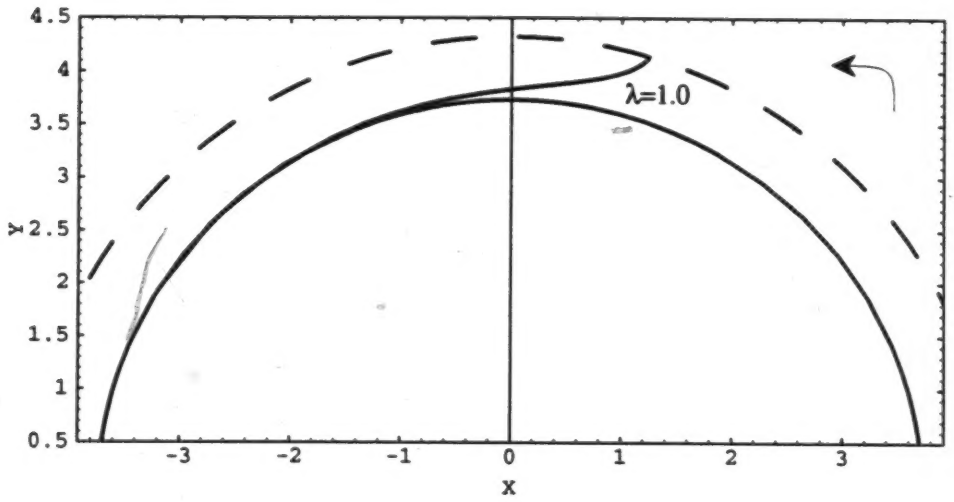

Figure 3.3.7: A typical string configuration where $\rho < \rho_1$ (Case 2(b)). The string is seen to spiral into the horizon $\rho = \rho_+$ from $\rho = \rho_1$ ($\alpha = 0.43, \omega = 0.04$).

the string configurations are similar to the one shown in figure 3.3.7.

Figures 3.3.4-3.3.8 illustrate the qualitatively distinct types of motion of rigidly rotating strings in the Kerr spacetime. In all figures the inner solid circle is the event horizon ρ_+. The dashed circle immediately outside the horizon is the circle $\rho = \rho_1$, and the outer dashed circle (if shown) is the circle $\rho = \rho_2$. String configurations (t =const slices) are shown by solid lines adjacent to the corresponding value λ of the angular momentum parameter.

For "subluminal" motion the apparent velocity is less than the velocity of light, while for "superluminal" motion the apparent velocity is greater than the velocity of light. In both cases the physical (orthogonal to the string) velocity is less than the velocity of light. We described the origin of this phenomenon in Section 3.3.

Besides these main categories of motion there are possible different boundary

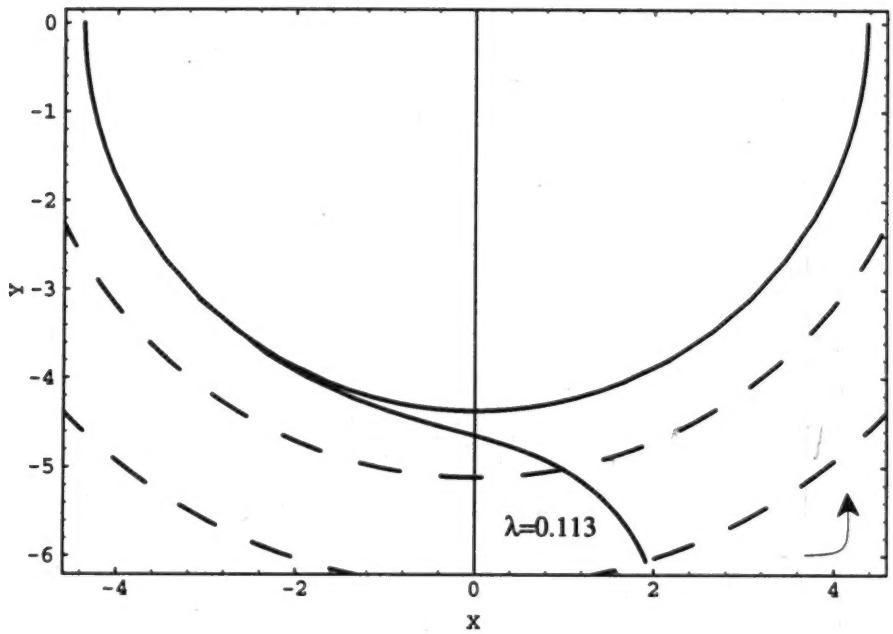

Figure 3.3.8: A string configuration where $\rho_0 = \rho_1$. The string is seen to pass through ρ_1 and to spiral into the horizon $\rho = \rho_+$ ($\alpha = 0.43$, $\omega = 0.04$).

cases when ρ_0 coincides either with ρ_1 or with ρ_2. These cases require special analysis. For special values of the parameters one might expect that a string passes through (and beyond) these points remaining regular and timelike (see figure 3.3.8 for example). We will examine the case of non-rotating strings that pass through the static limit in Chapter 4.

3.4 Gravitational Radiation from Trapped Strings

The string configurations with end points on the black hole horizon are of particular interest (figure 3.3.5). These configurations occur when either the Killing vector χ

has no zeroes, or when the string configuration lies within the cylinder, $r = r_0$, where r_0 is smallest zero of χ. The ends of the string lie on the horizon of the black hole. Physically, these cases represent trapped strings that are saved from being entirely swallowed by the conservation of angular momentum. In this section we will begin to investigate the gravitational radiation from such trapped strings. We do not treat this subject fully here but provide some ideas for further investigation.

The gravitational radiation from oscillating loops of cosmic strings has been studied in some detail. The power in gravitational radiation produced by an isolated loop of length ℓ, can be estimated using the quadrupole formula [33]:

$$P \propto cG\left(\dddot{I}_{ij}\right)^2, \tag{3.58}$$

where I_{ij} is the quadrupole moment of the string configuration. Dimensionally if ℓ is the only length scale, $I_{ij} = \mu\ell^3$, while each time derivative adds a factor of ℓ^{-3}. Thus we have

$$P \sim cG\mu^2. \tag{3.59}$$

For GUT strings, where $G\mu/c^2 \sim 10^{-6}$, we see that $P \sim 10^{47}$erg/ s. This is true for isolated loops, but would also seem to be a reasonable estimate for trapped loops of length ℓ rotating about a black hole. The signal from a trapped string however would be periodic and come from a single direction; the radiation from a network of strings would not appear to come from a single source and would not have a periodicity. There are other factors involved in the detection of such sources (for example, the likely number of sources in a given volume; see Thorne [34] for a recent review) which we do not consider here.

To study the situation further we propose studying closed string configurations in flat spacetime with a fixed point. Recall that in Minkowski spacetime the general string solution was given by

$$\mathbf{X} = \frac{1}{2}\left[\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)\right], \qquad \mathbf{a'}^2 = \mathbf{b'}^2 = 1. \tag{3.60}$$

If $\mathbf{X}(\sigma,t)$ is a closed loop with a fixed point then we impose boundary conditions $\mathbf{X}(0,t) = \mathbf{X}(\ell,t) = 0$ where $0 \leq \sigma \leq \ell$. Thus

$$\mathbf{b}(\sigma) = -\mathbf{a}(-\sigma) \qquad \mathbf{b}(\sigma+\ell) = \mathbf{b}(\sigma-\ell) \tag{3.61}$$

so that

$$\mathbf{X}(\sigma,t) = 1/2\,(\mathbf{b}(t+\sigma) - \mathbf{b}(t-\sigma))\,. \tag{3.62}$$

Further these solutions are all periodic in time with period ℓ:

$$\begin{aligned} \mathbf{X}(\sigma,t+\ell) &= 1/2\,(\mathbf{b}(t+\ell-\sigma) - \mathbf{b}(t+\ell-\sigma)) \\ &= 1/2\,(\mathbf{b}(t+\ell-\sigma) - \mathbf{b}(t-\ell-\sigma)) \\ &= \mathbf{X}(\sigma+\ell,t). \end{aligned} \tag{3.63}$$

We also note that, if no external force acts on the string, the centre of mass of such solutions is located at the origin

$$\int_0^\ell d\sigma \dot{\mathbf{X}} = 0, \tag{3.64}$$

and that $\mathbf{X}(\ell/2,t) = 0$. Thus such a solution may be described as a 'figure of eight' configuration rotating about its centre on the origin with period ℓ.

The power in gravitational radiation from a periodic source in the weak field limit (to lowest order in G) can be written (see Weinberg [33])

$$P = \sum_m \int d\Omega \frac{dP_m}{d\Omega}, \tag{3.65}$$

$$\frac{dP_m}{d\Omega} = \frac{G\omega_m^2}{\pi}\left[\tilde{T}^\star_{\mu\nu}(\omega_m,\mathbf{k})\tilde{T}^{\mu\nu}(\omega_m,\mathbf{k}) - \frac{1}{2}\left|\tilde{T}^\mu_\mu(\omega_m,\mathbf{k})\right|^2\right], \tag{3.66}$$

where $\tilde{T}^{\mu\nu}(\omega_m,\mathbf{k})$ is the Fourier transform of the stress-energy tensor. The string has a period ℓ, so the frequencies $\omega_m = 2\pi m/\ell$.

Garfinkle and Vachaspati [35] have written down the form of (3.66) for string solutions in flat spacetime. We follow their approach here for our fixed point solutions.

Working in the conformal gauge (equations 2.29) we have

$$\tilde{T}^{\mu\nu}(\omega_m, \mathbf{k}) = \frac{1}{\ell}\int_0^\ell dt \exp i\omega_m t \int d^3x \exp -i\mathbf{k}\cdot\mathbf{x} T^{\mu\nu}(\mathbf{X}, t) \tag{3.67}$$

$$T^{\mu\nu}(\mathbf{X}, t) = \mu\int_0^\ell d\sigma \left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\mu\right)\delta^{(3)}(\mathbf{x}-\mathbf{X}). \tag{3.68}$$

Note that because $b(\sigma)$ has period 2ℓ it is convenient formally to integrate around the loop twice while halving the mass per unit length of the string. We will do this in what follows.

Thus the Fourier transform can be expressed as

$$\begin{aligned}\tilde{T}^{\mu\nu}(\omega_m, \mathbf{k}) &= \frac{\mu}{4\ell}\int_0^{2\ell} dt \exp(i\omega_m t)\int_0^{2\ell} d\sigma \exp(-i\mathbf{k}\cdot\mathbf{X})\left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\mu\right)\\ &= \frac{\mu\ell}{4\pi^2}\int_0^{2\pi} d\eta \exp im(\eta - \frac{2\pi}{\ell}\mathbf{k}\cdot\mathbf{b}(\eta))\int_0^{2\pi} d\xi \exp -im(\xi + \frac{2\pi}{\ell}\mathbf{k}\cdot\mathbf{a}(\xi))\\ &\qquad \left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\mu\right)\end{aligned} \tag{3.69}$$

where $\eta = \pi(\sigma+t)/\ell$ and $\nu = \pi(\sigma-t)/\ell$. For the fixed point solutions (3.62) we have $a(\eta) = -b(-\eta)$ so

$$\begin{aligned}\tilde{T}^{\mu\nu}(\omega_m, \mathbf{k}) &= \frac{\mu\ell}{4\pi^2}\int_0^{2\pi} d\eta \exp im(\eta - \frac{2\pi}{\ell}\mathbf{k}\cdot\mathbf{b}(\eta))\int_0^{2\pi} d\xi \exp -im(\xi - \frac{2\pi}{\ell}\mathbf{k}\cdot\mathbf{b}(\xi))\\ &\qquad \left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\mu\right).\end{aligned} \tag{3.70}$$

It is convenient here to introduce the basis $k^\mu = 1/\sqrt{2}(1,\mathbf{k})$, $l^\mu = 1/\sqrt{2}(1,-\mathbf{k})$,; $n_2^\mu = (0,\mathbf{n}_2)$, $n_3^\mu = (0,\mathbf{n}_2)$ where the unit three vectors n_R $(R = 2,3)$ are orthogonal to $\mathbf{k}$. In this basis the metric can be written

$$\eta^{\mu\nu} = n_2^\mu n_2^\nu + n_3^\mu n_3^\nu - k^\mu l^\nu - l^\mu k^\nu. \tag{3.71}$$

Noting that integrals of the following form vanish

$$\mathbf{k}\cdot\int_0^{2\pi} d\eta \mathbf{b}'(\eta) exp\left(im(\eta - \frac{2\pi}{L}\mathbf{k}\cdot\mathbf{b})\right) = 0, \tag{3.72}$$

$$\mathbf{k}\cdot\int_0^{2\pi} d\xi \mathbf{b}'(\xi) exp\left(-im(\xi - \frac{2\pi}{L}\mathbf{k}\cdot\mathbf{b})\right) = 0 \tag{3.73}$$

we see that the power can be written solely in terms of the integrals

$$K^{\pm}_{m,R} = \frac{1}{2\pi}\mathbf{n}_R \cdot \int_0^{2\pi} \mathbf{b}'(\eta) exp\left(im(\eta \pm \frac{2\pi}{L}\mathbf{k}\cdot\mathbf{b})\right) \tag{3.74}$$

and their complex conjugates. Specifically for any fixed point solution described by $b(\sigma)$ we have

$$\frac{dP_m}{d\Omega} = 8\pi G\mu^2 m^2 \left[\left|K^+_{m,2}K^-_{m,2} - K^+_{m,3}K^-_{m,3}\right|^2 + \left|K^+_{m,2}K^-_{m,3} + K^+_{m,3}K^-_{m,2}\right|^2\right]. \tag{3.75}$$

We are interested in approximating the motion of configurations like that of figure 3.3.5. We can do so by cutting the 'figure of eight' type solution in half at the origin leaving a kink at the origin. This type of boundary condition mimics the effect of an external force, acting at the origin to 'trap' the string. In this case however (3.72-3.73) no longer vanish and the expression for the power is considerably more complicated. This will not be considered here.

Strings decay in a number of ways. One important consideration in the lifetime of a cosmic string loop is the formation of secondary loops when the string self-intersects [36]. This can rapidly diminish the size of an oscillating string. For an arbitrary solution $\mathbf{b}$ (which satisfies the constraint $\mathbf{b}'^2 = 1$) self-intersection will occur if the solution satisfies the three equations $\mathbf{b}(\tau_1+\sigma_1)-\mathbf{b}(-\sigma_1) = \mathbf{b}(\tau_1+\sigma_2)-\mathbf{b}(\tau_1-\sigma_2)$ for some distinct σ_1 and $\sigma_2 \in [0,\ell)$ at some τ_1. Thus it seems likely that such self-intersecting configurations do not dominate the solution space and that a typical string solution will be relatively long lived, so that the decay will be dominated by gravitational radiation.

As an illustration we will conclude with a computation of the power in gravitational radiation due to a rigidly rotating rod solution rotating about its centre in the xy plane [37]. This is a solution of (3.21-3.22) with $L = p = 0$ of length $\ell = 2/\Omega$ (the end-points lie on the cylinder $\rho = 1/\Omega$). Our formalism here requires that the solution $\mathbf{X}(t,\sigma)$ be closed rather than open; formally we will assume that the loop

is composed of two rigidly rotating strings with linear mass density $\mu/2$ lying along side each other.

In this case we find that the solution is given by

$$\mathbf{b}(\sigma) = \frac{\ell}{2\pi}\left(\sin(2\pi\sigma/\ell), \cos(2\pi\sigma/\ell), 0\right). \tag{3.76}$$

Clearly this satisfies the constraint (3.60). It is convenient to work in spherical coordinates (r, θ, ϕ) where $\mathbf{k}$ is a unit vector in the radial direction and the orthognal vectors $\mathbf{n}_R$ are the unit vectors in the θ and ϕ directions respectively. In this case the integrals $K^{\pm}_{m,R}$ are linear combinations of Bessel functions of the first kind:

$$\begin{aligned} K^{+}_{m,\theta} &= ie^{im\phi} J'_m(m\sin\theta), \\ K^{+}_{m,\phi} &= e^{im\phi}\cot\theta J_m(m\sin\theta). \end{aligned} \tag{3.77}$$

where $K^{+}_{m,\theta} = (-1)^{m+1} K^{-}_{m,\theta}$. Thus we find that the power radiated can be written:

$$\frac{dP_m}{d\Omega} = 8\pi G\mu^2 m^2 \left[J'^{\,4}_m(m\sin\theta) + 6\cot^2\theta J'^{\,2}_m(m\sin\theta) J^2_m(m\sin\theta) + \cot^4\theta J^4_m(m\sin\theta) \right].$$

For this solution the total power P diverges since $P_m \sim 1/m$ for large m; physically this is because the end-points of the string are moving at the speed of light for the entire motion of the string. This is not necessarily a problem because there is evidence to suggest a natural cut-off for these higher oscillation modes [38].

Chapter 4

Stationary Strings in the Kerr-Newman Spacetime

Black hole solutions in a spacetime of fewer than 4 dimensions have been discussed for a long time (see for example [39] and references therein). Such solutions are of interest mainly because they provide toy models which allow one to investigate unsolved problems in four-dimensional black hole physics. The interest in 2-D black holes greatly increased after Witten [40] and Mandal, Sengupta, and Wadia [41] were able to show that 2-D black hole solutions naturally arise in superstring motivated 2-D dilaton gravity. For reviews of many aspects of 2-D black hole physics and its relation to 4-D gravity see [42], [43] and [44]. The main purpose of this chapter is to show that objects which behave as 2-D black holes may be physically possible. Namely, we consider a cosmic string interacting with a usual 4-D stationary black hole.

We will study the stationary solutions $\Omega = 0$ in the Kerr-Newman spacetime from Section 3.3.2. These represent the stationary final states of a captured infinite string, with end-points fixed at infinity. We show that there is only a very special family of solutions describing a stationary string which enters the ergosphere, namely the strings lying on cones of a given angle θ =const. We will demonstrate that the induced 2-D geometry of a stationary string crossing the static limit surface and entering the ergosphere of a rotating black hole has the metric of a 2-D black or white hole. The horizon of such a 2-D string hole coincides with the intersection of the string world-sheet with the static limit surface. We shall also demonstrate that the 2-D string hole geometry can be tested by studying the propagation of string

perturbations. The perturbations propagating along the cone strings (θ =const) are shown to obey the relativistic equations for a coupled system of two scalar fields. These results generalize the results of [15] where the corresponding equations were obtained and investigated for strings lying in the equatorial plane. The quantum radiation of string excitations (stringons) and the thermodynamic properties of string holes are discussed. The remarkable property of 2-D string holes as physical objects is that besides quanta (stringons) living and propagating only on the 2-D world-sheet there exist other field quanta (gravitons, photons etc.) living and propagating in the surrounding physical 4-D spacetime. Such quanta can enter the ergosphere as well as leave it and return to the exterior. For this reason, the presence of the extra physical dimensions enables dynamical interaction between the interior and exterior of a 2-D string black hole to occcur. This interaction appears acausal from the perspective of the internal 2-D geometry.

The chapter is organized as follows: In Section 2 we collect results concerning the Kerr-Newman geometry which are necessary for the following sections. In Section 3 we introduce the notion of a principal Killing surface and we prove that a principal Killing surface is a minimal 2-surface embedded in the 4-dimensional spacetime. In section 4 we prove a uniqueness theorem, i.e. we prove the statement that the principal Killing surfaces are the only stationary minimal 2-surfaces that are timelike and regular in the vicinity of the static limit surface of the Kerr-Newman black hole. In Section 4 we also relate the principal Killing surfaces to the world-sheets of a particular class of stationary cosmic strings: the cone strings of section 3.3.2. In Section 5 we show that the internal geometry of these world-sheets is that of a two-dimensional black or white hole, and we discuss the geometry of such string holes. In Section 6 we consider the propagation of perturbations along a stationary string using the covariant approach developed in [12] which was detailed in Section 2.3. We show that the corresponding equations coincide with a system of coupled equations

for a pair of scalar fields on the two-dimensional string hole background. In Section 7, we discuss some of the physics of string holes. Finally, in the Appendix we show that these 2-D string black holes can be obtained as solutions of 2D-dilaton gravity.

4.1 Killing vectors in the Kerr-Newman geometry

The Kerr-Newman metric (3.31) is of Petrov type D and possesses two principal null directions l_+^μ and l_-^μ. Each of these null vectors obeys the relation:

$$C^{(+)}_{\alpha\beta\gamma\delta} l^\beta l^\delta = C l_\alpha l_\gamma, \tag{4.1}$$

where:

$$C^{(+)}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} + iC^*_{\alpha\beta\gamma\delta}, \qquad C^*_{\alpha\beta\gamma\delta} = 1/2 e_{\alpha\beta\mu\nu} C^{\mu\nu}{}_{\gamma\delta}. \tag{4.2}$$

Here $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $e_{\alpha\beta\mu\nu}$ is the totally antisymmetric tensor, and $C_\pm$ are non-vanishing complex numbers. The Goldberg-Sachs theorem [45] implies that the integral lines $x^\mu(\lambda)$ of principal null directions

$$\frac{dx^\mu}{d\lambda_\pm} = \mp l_\pm^\mu \tag{4.3}$$

are null geodesics ($l^\mu l_\mu = 0, \quad l^\mu l^\nu{}_{;\mu} = 0$) and their congruence is shear free. We denote by γ_+ and γ_- ingoing and outgoing principal null geodesics, respectively, and choose the parameter $\lambda_\pm$ to be an affine parameter along the geodesic.

Consider the Killing vector $\xi_{(t)}$ in the Kerr-Newman spacetime (3.31). We will drop the subscript (t) and denote this Killing vector by ξ in this chapter (Note that the results in this chapter do not apply to the Killing vector $\xi_{(\phi)}$). The Killing equation implies that the tensor $\xi_{\mu;\nu}$ is antisymmetric, and its eigenvectors with non-vanishing eigenvalues are null. In the Kerr-Newman geometry (3.31) $\xi_{\mu;\nu}$ is of the form:

$$\xi_{\mu;\nu} = (\Delta F'/2\Sigma) l_{+[\mu} l_{-\nu]} + (2ia(1-F)\cos\theta/\Sigma) m_{[\mu} \bar{m}_{\nu]}, \tag{4.4}$$

where we have made use of the complex null vectors, m and $\bar{m}$, that complete the Kinnersley null tetrad (here $l^\mu_\pm m_\mu = 0$). In Boyer-Lindquist coordinates and in the normalization where $l^\mu_+ l_{-\mu} = -2\Sigma/\Delta$ and $m^\mu \bar{m}_\mu = 1$, the tetrad can be explicitly written as

$$\begin{aligned} l^\mu_\pm &= \left((r^2+a^2)/\Delta, \mp 1, 0, a/\Delta\right), \\ m^\mu &= \frac{1}{\sqrt{2}(r+ia\cos\theta)}(ia\sin\theta, 0, 1, i/\sin\theta). \end{aligned} \tag{4.5}$$

An interesting property of the Kerr-Newman geometry, as can be seen from (4.4), is that the principal null vectors $l_\pm$ of the Weyl tensor are eigenvectors of $\xi_{\mu;\nu}$. Namely one has:

$$\xi_{\mu;\nu} l^\nu_\pm = \mp\kappa l_{\pm\mu}, \qquad \kappa = \pm\frac{1}{2} l^\nu_\pm (\xi^2)_{;\nu} = \frac{1}{2}F_{,r} = \frac{Mr^2 - rQ^2 - Ma^2\cos^2\theta}{\Sigma^2}. \tag{4.6}$$

These equations, (4.4) and (4.6), will play an important role later in our analysis.

Notice also that the electromagnetic field tensor $\mathbf{F}$ has the form:

$$F_{\mu\nu} = -\frac{\Delta}{\Sigma}\left(\frac{Q(r^2 - a^2\cos^2\theta)}{\Sigma^2}\right) l_{+[\mu} l_{-\nu]} + \frac{4iQar\cos\theta}{\rho^4} m_{[\mu}\bar{m}_{\nu]}, \tag{4.7}$$

so that:

$$F_{\mu\nu} l^\nu_\pm = \mp\frac{Q(r^2 - a^2\cos^2\theta)}{\rho^4} l_{\mu\pm}. \tag{4.8}$$

4.2 Principal Killing surfaces

Our aim is to consider stationary, non-rotating configurations of cosmic strings in the gravitational field of a charged rotating black hole (namely, stationary configurations in the Kerr-Newman spacetime 3.31). In particular, we are interested in the situation when a string is trapped by a black hole; that is when the string crosses the black hole's static limit surface and enters the ergosphere. The mathematical problem we wish to solve, is that of finding stationary timelike minimal surfaces which intersect the static limit surface of a rotating black hole. (Recall from chapter 2 that Nambu-Goto world-sheets are minimal surfaces.)

For this purpose we begin by reviewing the general properties of stationary timelike surfaces. Let S be a two-dimensional timelike surface embedded in a stationary spacetime, and let ξ be the corresponding Killing vector which is timelike at infinity. In the previous chapter we considered stationary, axisymmetric spacetimes; here we merely require that the spacetime be stationary. Such a surface S is said to be stationary if it is everywhere tangent to the Killing vector field ξ. In Chapter 3, we considered surfaces in a stationary, axisymmetric spacetime that were tangent to the Killing vector $\xi_{(t)} + \Omega\xi_{\phi}$; we now set $\Omega = 0$. For any such surface S, there exists two linearly independent null vector fields l, tangent to S. We assume that the integral curves of l form a congruence and cover S (i.e. each point $p \in S$ lies on exactly one of these integral curves). In other words, the surface S is an integral submanifold of the two null vector fields (see page 20).

Thus we can construct a stationary timelike surface S in the following way: consider a null ray γ with tangent vector field l such that $\xi \cdot l$ is non-vanishing everywhere along γ. There is precisely one Killing trajectory with tangent vector ξ that passes throughеach point $p \in \gamma$. This set of Killing trajectories passing through γ forms a stationary 2-D surface S (see figure 4.2.1). We define l over S by Lie propagation along each Killing trajectory; that is, we define l along each Killing trajectory by $\mathcal{L}_{\xi} l = 0$. We call γ a basic ray of S. It is easily verified that l remains null when defined in this manner over S.

We can use the Killing time parameter u, and the affine parameter λ along γ as coordinates on S (since $\mathcal{L}_{\xi} l = 0$ on S, the two vector fields are coordinate vector fields on S). In these coordinates $\zeta^A = (u, \lambda)$ one has $x^{\mu}_{,0} = \xi^{\mu}$ and $x^{\mu}_{,1} = l^{\mu}$ and the induced metric $G_{AB} = g_{\mu\nu} x^{\mu}_{,A} x^{\nu}_{,B}$ $(A, B, ... = 0, 1)$ is of the form:

$$dS^2 = G_{AB} d\zeta^A d\zeta^B = -F du^2 + 2(\xi \cdot l) du d\lambda. \tag{4.9}$$

In the case of a black hole the Killing vector ξ becomes null at the static limit surface S_{st}. In what follows we always choose l to be that of the two possible null vector fields on S which does not coincide with ξ on the static limit surface S_{st}. In this case the metric (4.9) is regular at S_{st}.

Recall that the condition that a surface S is minimal can be written in terms

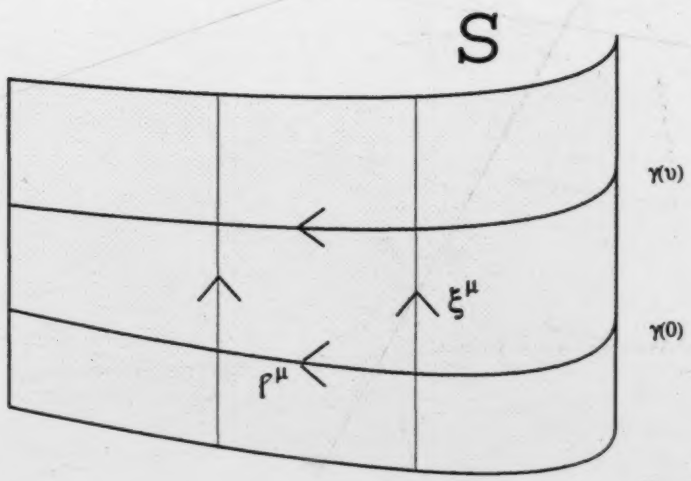

Figure 4.2.1: The figure illustrates the construction of a stationary world-sheet. The null ray γ has tangent vector field l. There is precisely one Killing trajectory with tangent vector ξ that passes through each point $p \in \gamma$. This set of Killing trajectories passing through γ forms a stationary 2-D surface S.

of the trace of the second fundamental form (2.23) as follows:

$$K_{RA}{}^{A} \equiv G^{AB} K_{RAB} = 0. \tag{4.10}$$

We find that in the metric (4.9) the second fundamental form is given by:

$$\begin{aligned} K_{RA}{}^{A} &= g_{\mu\nu} G^{AB} n_R^{\mu} x^{\gamma}_{,A} \nabla_{\gamma} x^{\nu}_{,B} \\ &= g_{\mu\nu} n_R^{\mu} \left(\frac{2}{(\xi \cdot l)} l^{\gamma} \nabla_{\gamma} \xi^{\nu} + \frac{F}{(\xi \cdot l)^2} l^{\gamma} \nabla_{\gamma} l^{\nu} \right). \end{aligned} \tag{4.11}$$

Consider a special type of stationary timelike 2-surface in the Kerr-Newman geometry. Namely, a surface for which the null vector l coincides with one of the principal null geodesics $l_{\pm}$ of the Kerr-Newman geometry. One can verify that $l_{\pm}$ and ξ are surface-forming in the Kerr-Newman goemetry (in particular, see Chapter 5). We call such a surface $S_{\pm}$ a principal Killing surface and $\gamma_{\pm}$ its basic ray. We shall use indices $\pm$ to distinguish between quantities connected with $S_{\pm}$. The fact that $l_{\pm}$ are geodesics ensures that $l_{\pm}^{\gamma} \nabla_{\gamma} l_{\pm}^{\mu} \propto l_{\pm}^{\mu}$. In addition, from equation (4.6), $l_{\pm}^{\gamma} \nabla_{\gamma} \xi^{\mu} \propto l_{\pm}^{\mu}$ which, because of the contraction with n_R^{ν}, guarantees that $K_{RA}{}^{A}$ vanishes for a principal Killing surface, i.e. every principal Killing surface is minimal. Thus $S_{\pm}$ are stationary solutions of the Nambu-Goto equations.

It should be stressed that the principal Killing surfaces are only very special stationary minimal surfaces. A principal Killing surface is uniquely determined by indicating two coordinates (angles) of a point where it crosses the static limit surface. Because of the axial symmetry only one of these two parameters is non-trivial. The general stationary string solution in the Kerr-Newman spacetime depends on 3 parameters (2 of which are non-trivial) as we saw in section 3.3.2.

4.3 Uniqueness Theorem for Principal Killing Surfaces

We prove now that the only stationary timelike minimal 2-surfaces that cross the static limit surface S_{st} and are regular in its vicinity are the principal Killing surfaces.

Consider a stationary timelike surface S described by the line element (4.9). Using the completeness relation (2.5) and the metric (4.9) we obtain:

$$K^2 = z^\mu z_\mu, \qquad z^\mu \equiv \frac{2}{(\xi \cdot l)} l^\gamma \nabla_\gamma \xi^\mu + \frac{F}{(\xi \cdot l)^2} l^\gamma \nabla_\gamma l^\nu. \tag{4.12}$$

In other words S is minimal if and only if z^μ is null so that $K^2 = 0$ (clearly if S is a principal Killing surface then $z^\mu \propto l^\mu_\pm$ and this condition is satisfied). In general we observe that $l \cdot z$ vanishes, as l^μ is null and $\xi_{\mu;\nu}$ is antisymmetric. Thus if z^μ is null then it must be proportional to l^μ. The condition that $K^2 = 0$ in the line element (4.9) then becomes:

$$2(\xi \cdot l) l^\rho \nabla_\rho \xi^\mu + F l^\rho \nabla_\rho l^\mu + (\xi \cdot l) l^\rho \frac{d}{dx^\rho} \left(\frac{F}{\xi \cdot l} \right) l^\mu = 0. \tag{4.13}$$

It is easily verifed that equation (4.13) is invariant under reparameterizations of l^μ, i.e. if l^μ satisfies (4.13) then so does $g(x) l^\mu$. Thus without loss of generality we may normalize l^μ so that $l \cdot \xi = -1$. Then (4.13) becomes:

$$-2 l^\rho \nabla_\rho \xi^\mu + F l^\rho \nabla_\rho l^\mu + l^\rho \frac{dF}{dx^\rho} l^\mu = 0. \tag{4.14}$$

Since $l^\rho \nabla_\rho l^\mu$ is regular on S, this equation at the static limit surface ($F = 0$) reduces to:

$$(\xi_{\mu;\rho} - \frac{1}{2} \frac{dF}{dx^\rho} l_\mu) l^\rho = 0, \tag{4.15}$$

that is, l^ρ is a real eigenvector of $\xi_{\mu;\rho}$. From equation (4.4) follows that the only real eigenvectors of $\xi_{\mu;\rho}$ are colinear with either l_+ or l_-. Thus we have $l \propto l_\pm$ at the static limit surface.

Now suppose there exists a timelike minimal surface S different from $S_\pm$. At the static limit surface S_{st}, we must have $l \propto l_+$ (or $l \propto l_-$). We will consider the case where $l \propto l_+$. In the vicinity of the static limit surface, since l obeys (4.15), l can have only small deviations from l_+. From the conditions $l \cdot l = 0$ and $l \cdot \xi = -1$, we then get the following general form of l in the vicinity of the static limit surface:

$$l = [1 + \frac{ia \sin\theta}{\sqrt{2}\rho}(B - \bar{B})] l_+ + \bar{B} m + B \bar{m} + \mathcal{O}(B^2), \tag{4.16}$$

up to first order in $(B, \bar{B})$. We then insert this expression into (4.13), contract by $\bar{m}_\mu$ and keep only terms linear in $(B, \bar{B})$:

$$-2\bar{m}_\mu l^\rho \nabla_\rho \xi^\mu = \frac{-2ia(1-F)\cos\theta}{\Sigma}\bar{B} + \mathcal{O}(B^2), \tag{4.17}$$

$$\bar{m}_\mu l^\rho \frac{dF}{dx^\rho} l^\mu = l_+^\rho \frac{dF}{dx^\rho}\bar{B} + \mathcal{O}(B^2) = -F'\bar{B} + \mathcal{O}(B^2), \tag{4.18}$$

$$\begin{aligned}\bar{m}_\mu F l^\rho \nabla_\rho l^\mu &= F l_+^\rho \frac{d\bar{B}}{dx^\rho} - 2F l_+^\mu m^\rho \bar{m}_{(\rho;\mu)}\bar{B} + F\bar{m}^\rho \bar{m}_\mu l^\mu_{+;\rho} B + \mathcal{O}(B^2) \\ &= -F\frac{d\bar{B}}{dr} - \frac{(\sqrt{\Sigma} + 2ia\cos\theta)F}{\Sigma}\bar{B} + \mathcal{O}(B^2),\end{aligned} \tag{4.19}$$

where the last equality was obtained by direct calculation using (2.8), (2.11). Thus altogether:

$$F\frac{d\bar{B}}{dr} = -\bar{B}[\frac{dF}{dr} + \frac{2ia\cos\theta}{\Sigma} + \frac{F}{\sqrt{\Sigma}}] + \mathcal{O}(B^2). \tag{4.20}$$

It is convenient to rewrite this equation in the form:

$$\frac{d\bar{B}}{dr^*} = -\omega\bar{B}; \qquad \omega \equiv \frac{dF}{dr} + \frac{2ia\cos\theta}{\Sigma} + \frac{F}{\rho} \tag{4.21}$$

and we have introduced the tortoise-coordinate r^* defined by:

$$\frac{dr}{dr^*} = F(r). \tag{4.22}$$

Near the static limit surface the complex frequency ω is given by:

$$\omega = \frac{2(r_{st} - M + ia\cos\theta)}{r_{st}^2 + a^2\cos^2\theta} + \mathcal{O}(r - r_{st}) \equiv \omega_{st} + \mathcal{O}(r - r_{st}) \tag{4.23}$$

The solution of equation (4.21) near the static limit surface is then given by:

$$\bar{B} = ce^{-\omega_{st}r^*}; \qquad c = \text{const.} \tag{4.24}$$

Notice that $Re(\omega_{st}) > 0$, thus $\bar{B}$ is oscillating with infinitely growing amplitude near the static limit surface. A solution regular near the static limit surface ($r^* \to -\infty$) can therefore only be obtained for $c = 0$, which implies that $B = \bar{B} = 0$. The argument is similar, with the same conlcusion, in the case where $l \propto l_-$ at the static limit surface. Thus we have shown that S is minimal if and only if $l \propto l_\pm$. This proves the uniqueness theorem: The only stationary timelike minimal 2-surfaces that

cross the static limit surface S_{st} and are regular in its vicinity are the principal Killing surfaces.

We now discuss the physical meaning of this result. For this purpose it is convenient to introduce the ingoing (+) and outgoing (−) Eddington-Finkelstein coordinates $(u_\pm, \tilde{\phi}_\pm)$:

$$du_\pm = dt \pm \Delta^{-1}(r^2 + a^2)dr, \qquad d\tilde{\phi}_\pm = d\phi \pm \Delta^{-1}adr, \tag{4.25}$$

and to rewrite the Boyer-Lindquist metric (3.31) as:

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\Sigma}[du_\pm - a\sin^2\theta d\tilde{\phi}_\pm]^2 + \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)d\tilde{\phi}_\pm - adu_\pm]^2 \\ &+ \Sigma d\theta^2 \pm 2dr[du_\pm - a\sin^2\theta d\tilde{\phi}_\pm]. \end{aligned} \tag{4.26}$$

The electromagnetic field tensor is:

$$\begin{aligned} \mathbf{F} = &\frac{Q(r^2 - a^2\cos^2\theta)}{\Sigma^2}\mathbf{d}r \wedge [\mathbf{d}u_\pm - a\sin^2\theta\mathbf{d}\tilde{\phi}_\pm] \\ &+ \frac{2arQ\cos\theta\sin\theta}{\Sigma^2}\mathbf{d}\theta \wedge [(r^2+a^2)\mathbf{d}\tilde{\phi}_\pm - a\mathbf{d}u_\pm]. \end{aligned} \tag{4.27}$$

We have shown that any stationary minimal 2-surface that crosses the static limit must have $x^\mu_{,1} \propto l^\mu_\pm$. Using the explicit form of $l_\pm$ in Boyer-Lindquist coordinates (4.5), we can choose the affine parameter along $\gamma_\pm$ to coincide with r such that $x' = \mp l_\pm$, where the prime denotes derivative with respect to r. We can then read off θ' and ϕ' for these surfaces $S_\pm$:

$$\theta' = 0, \qquad \tilde{\phi}'_\pm = 0. \tag{4.28}$$

In the Eddington-Finkelstein coordinates, the induced metric on the principle Killing surface, $S_\pm$, is then:

$$dS^2 = -Fdu_\pm^2 \pm 2drdu_\pm; \qquad F = 1 - \frac{2Mr - Q^2}{\Sigma}. \tag{4.29}$$

The induced electromagnetic field tensor is:

$$\mathbf{F} = \frac{Q}{\Sigma^2}(r^2 - a^2\cos^2\theta)\mathbf{d}r \wedge \mathbf{d}u_\pm. \tag{4.30}$$

That is, the induced electric field is:

$$E_r = \frac{Q}{\Sigma^2}(r^2 - a^2\cos^2\theta). \tag{4.31}$$

Equations (4.28) imply that a principal Killing string configuration is located at the cone surface θ = const. The principal Killing strings are the cone string solutions noted in chapter 3. These cone strings are thus the only stationary world-sheets that can cross the static limit surface and are timelike and regular in its vicinity. Recalling the equations that describe the cone string configurations (3.46),

$$\tilde{\phi}_{\pm} = \text{const.}, \qquad q^2 = 2a|L|, \qquad \sin^2\theta = \text{const.} = |L|/a, \tag{4.32}$$

we find that they are a two-parameter family of solutions (notice however that due to the axial symmetry, only one of these parameters L is non-trivial). Physically it means that a stationary cosmic string can only enter the ergosphere in very special ways, corresponding to the angles (4.32).

4.4 Geometry of 2-D string holes

The metric (4.29) for S_+ describes a black hole, while for S_- it describes a white hole. For $a = 0$, $S_{\pm}$ are geodesic surfaces in the 4-D spacetime and they describe two branches of a geodesically complete 2-D manifold. However, it should be stressed that for the generic Kerr-Newman geometry ($a \neq 0$), only one of two basic null rays of the principal Killing surface, namely the ray $\gamma_{\pm}$ with tangent vector $l\pm$, is geodesic in the four-dimensional embedding space. The other basic null ray is geodesic in $S_{\pm}$ but not in the embedding space. This implies that in general (when $a \neq 0$) the principal Killing surface is not geodesic. Furthermore, it can be shown that the surfaces $S_{\pm}$, considered as 2-D manifolds, are geodesically incomplete with respect to the null geodesic γ'. Because the surfaces $S_{\pm}$ are not geodesic (when $a \neq 0$), we shall be able to show that it is possible to send causal signals from the inside of the 2-D black hole

to the outside of the 2-D black hole by exploiting the two extra dimensions of the 4-D spacetime.

It is evident that there exist causal lines leaving the ergosphere and entering the black hole exterior. This means that the "interior" and "exterior" of the 2-D black hole can be connected by 4-D causal lines. We show now that (at least for the points lying close to the static limit surface) the causal line can be chosen as a null geodesic. Consider for simplicity the stationary string corresponding to ($\theta = \pi/2,\ \tilde{\phi}_+ = 0$) and crossing the static limit surface in the equatorial plane of a Kerr black hole. We will demonstrate that there exists an outgoing null geodesic in the 4-D spacetime connecting the point $(r, \tilde{\phi}_+) = (2M - \epsilon, 0)$ of the cosmic string inside the ergosphere with the point $(r, \tilde{\phi}_+) = (2M + \epsilon, 0)$ of the cosmic string outside the ergosphere, for ϵ small. An outgoing null geodesic, corresponding to positive energy at infinity E and angular momentum at infinity L_z in the equatorial plane of the Kerr black hole background, is determined by [49]:

$$r^2 \frac{dr}{d\lambda} = \mathcal{P}, \tag{4.33}$$

$$r^2 \frac{du_+}{d\lambda} = -a\mathcal{U} + \frac{r^2 + a^2}{\Delta}[\mathcal{P} + \mathcal{Q}], \tag{4.34}$$

$$r^2 \frac{d\tilde{\phi}_+}{d\lambda} = -\mathcal{U} + \frac{a}{\Delta}[\mathcal{P} + \mathcal{Q}], \tag{4.35}$$

where:

$$\mathcal{U} \equiv aE - L_z, \qquad \mathcal{Q} \equiv Er^2 + a\mathcal{U}, \qquad \mathcal{P}^2 \equiv \mathcal{Q}^2 - \Delta\mathcal{U}^2. \tag{4.36}$$

We consider the case where $dr/d\lambda > 0$.

Inside the ergosphere the 4-D geodesic must follow the rotation of the black hole because of the dragging effect, that is, $d\tilde{\phi}_+/d\lambda > 0$ (for $a > 0$). However, after leaving the ergosphere the geodesic can reach a turning point in $\tilde{\phi}_+$ and then return ($d\tilde{\phi}_+/d\lambda < 0$) towards the cosmic string outside the static limit surface. To be more precise: provided $-L_z > aE$, there will be a turning point in $\tilde{\phi}_+$ outside the static limit surface at $r = r_0$:

$$r_0 = \frac{2M(aE - L_z)}{-L_z - aE} > 2M. \tag{4.37}$$

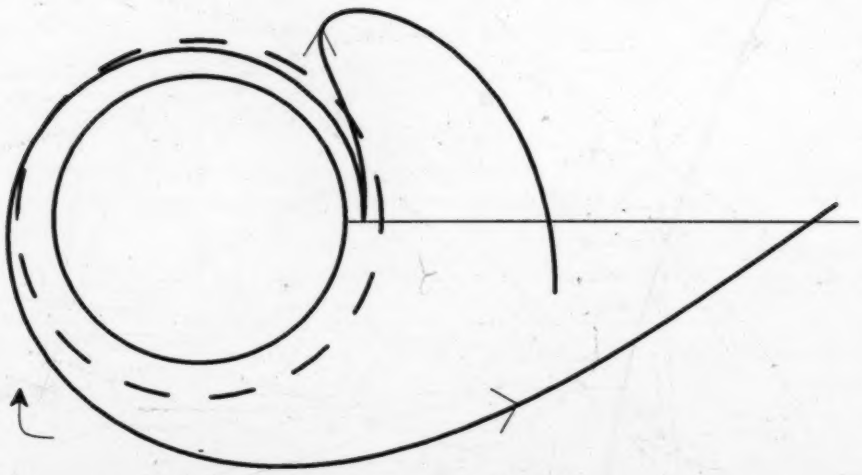

Figure 4.4.1: The figure shows two null trajectories that have an intersection with the string in the ergosphere and an intersection with the string outside the ergosphere. In Eddington-Finkelstein coordinates the cone string solution lies horizontally along the positive x-axis. The static limit is denoted by a dashed line and the horizon is indicated by a solid line. The string and trajectories lie in the equatorial plane of a Kerr black hole ($a/M = 0.8$).

Obviously the turning point in $\tilde{\phi}_+$ can be put at any value of r outside the static limit surface. If we choose E and L_z such that:

$$r_0 = 2M + \epsilon - \frac{M}{2a^2}\epsilon^2, \tag{4.38}$$

then, after reaching the turning point in $\tilde{\phi}_+$, the geodesic will continue in the direction opposite to the rotation of the 4-D black hole with constant $r = 2M + \epsilon$ (to first order in ϵ) and eventually reach the point $(r, \tilde{\phi}_+) = (2M + \epsilon, 0)$ of the cosmic string outside the ergosphere.

Equations (4.33-4.35) can be solved numerically. In figure 4.4.1 we show the trajectories of null geodesic rays that have an intersection with the string (lying horizontally on the x-axis) in the ergosphere and an intersection with the string outside the ergosphere. Note the ray that wraps once around the black hole before reintersecting the string. The trajactories and the string shown lie in the equatorial plane of a Kerr black hole.

4.5 String perturbation propagation

Recall the equations of motion for perturbations $\delta X^\mu = \Phi^R n_R^\mu$ on a background Nambu-Goto string world-sheet (see section 2.3):

$$\left\{\delta_{RS}\,\Box + 2\mu_{RS}{}^A\partial_A + \nabla_A\mu_{RS}{}^A - \mu_R{}^{TA}\mu_{STA} + \mathcal{V}_{RS}\right\}\Phi^S = 0. \tag{4.39}$$

We note that the perturbations (2.25) and the effective action (2.26) are invariant under rotations of the normal vectors, i.e. they are invariant under the transformations $n_R \mapsto \tilde{n}_R = \Lambda_R{}^S n_S$, $\Phi^R \to \tilde{\Phi}^R = \Lambda^R{}_S\Phi^S$, where:

$$[\,\Lambda\,]_R{}^S = \begin{pmatrix} \cos\Psi & -\sin\Psi \\ \sin\Psi & \cos\Psi \end{pmatrix}, \tag{4.40}$$

for some arbitrary real function Ψ. Thus we have a 'gauge' freedom in our choice of normal vectors.

Consider the scalar potential $\mathcal{V}_{RS} \equiv K_{RAB}K_S{}^{AB} - G^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho_R n^\sigma_S$. It is easily verified that the first term $K_{RAB}K_S{}^{AB}$ vanishes for the principal Killing surface $S_\pm$ independently of any choice of normal vectors n^ρ_R. We will also show that the second term on the right hand side is invariant under rotations of the vectors n_R, i.e. gauge invariant, in the Kerr-Newman spacetime.

Let $M = (n_2 + in_3)/\sqrt{2}$ where $\{n_2, n_3\}$ span the two-dimensional vector space normal to the cone string world-sheet. Then, under the rotation specified by (4.40) $M^\mu \mapsto \tilde{M}^\mu = e^{i\Psi}M^\mu$. We note that the combination $M^\mu\bar{M}^\nu$ is invariant under this transformation.

We will make use of the following equalities:

$$M^\mu\bar{M}^\nu = 1/2\,(n_2^\mu n_2^\nu + n_3^\mu n_3^\nu) - i/2\,(n_2^\mu n_3^\nu - n_3^\mu n_2^\nu), \tag{4.41}$$

$$M^\mu M^\nu = 1/2\,(n_2^\mu n_2^\nu - n_3^\mu n_3^\nu) + i/2\,(n_2^\mu n_3^\nu + n_3^\mu n_2^\nu). \tag{4.42}$$

Now consider:

$$\begin{aligned} G^{AB}x^\mu_{,A}x^\nu_{,B}R_{\mu\rho\sigma\nu}n^\rho_R n^\sigma_S &= (g^{\mu\nu} - \delta^{TQ}n_T^\mu n_Q^\nu)R_{\mu\rho\sigma\nu}n^\rho_R n^\sigma_S \\ &= -R_{\rho\sigma}n^\rho_R n^\sigma_S - \delta^{TQ}R_{\mu\rho\sigma\nu}n_T^\mu n_Q^\nu n^\rho_R n^\sigma_S. \end{aligned} \tag{4.43}$$

The second term on the right hand side can be written as:

$$\begin{aligned}\delta^{TQ}R_{\mu\rho\sigma\nu}n_T^\mu n_Q^\nu n_R^\rho n_S^\sigma &= (n_2^\mu n_2^\nu + n_3^\mu n_3^\nu)R_{\mu\rho\sigma\nu}n_R^\rho n_S^\sigma \\ &= \delta^{RS}R_{\mu\rho\sigma\nu}n_2^\mu n_2^\nu n_3^\rho n_3^\sigma \\ &= -\delta^{RS}R_{\mu\rho\sigma\nu}M^\mu M^\nu \bar{M}^\rho \bar{M}^\sigma, \end{aligned} \tag{4.44}$$

making use of (4.41) and the symmetries of the Riemann tensor only. This form is explicitly gauge invariant in any spacetime geometry.

It remains to verify that the term $R_{\rho\sigma}n_R^\rho n_S^\sigma$ is also gauge invariant. We note that M and the complex null vector m of the Kinnersley tetrad are related by the null rotation $M = m + E\,l$. We may then use the fact that m and $l_\pm$ are eigenvectors of $R_{\rho\sigma}$ (see equation (6.8)) to show:

$$R_{\rho\sigma}M^\rho M^\sigma = R_{\rho\sigma}\left(m^\rho m^\sigma + 2Em^\rho l_\pm^\sigma + E^2 l_\pm^\rho l_\pm^\sigma\right) = 0, \tag{4.45}$$

Notice that this holds in any gauge as $M^\rho M^\sigma \mapsto \tilde{M}^\rho \tilde{M}^\sigma = e^{2i\Psi}M^\rho M^\sigma$. Thus equating real and imaginary parts of $R_{\rho\sigma}M^\rho M^\sigma$ to zero one finds:

$$R_{\rho\sigma}n_2^\rho n_2^\sigma = R_{\rho\sigma}n_3^\rho n_3^\sigma, \qquad R_{\rho\sigma}n_2^\rho n_3^\sigma = -R_{\rho\sigma}n_3^\rho n_2^\sigma = 0. \tag{4.46}$$

Thus under a gauge transformation, we find that:

$$\begin{aligned} R_{\rho\sigma}\tilde{n}_2^\rho \tilde{n}_3^\sigma &= R_{\rho\sigma}(\cos\Psi n_2^\rho - \sin\Psi n_3^\rho)(\sin\Psi n_2^\sigma + \cos\Psi n_3^\sigma) \\ &= 0. \end{aligned} \tag{4.47}$$

It then follows that:

$$\begin{aligned} R_{\rho\sigma}\tilde{n}_2^\rho \tilde{n}_2^\sigma &= R_{\rho\sigma}(\cos\Psi n_2^\rho - \sin\Psi n_3^\rho)(\cos\Psi n_2^\sigma - \sin\Psi n_3^\sigma) \\ &= R_{\rho\sigma}n_2^\rho n_2^\sigma. \end{aligned} \tag{4.48}$$

Similarly $R_{\rho\sigma}n_3^\rho n_3^\sigma$ remains unchanged under rotation. Thus we conclude that $\mathcal{V}_{RS}$ is gauge invariant as $K_{RAB}K_S{}^{AB}$ vanishes independently of gauge in the Kerr-Newman spacetime.

The symmetry and gauge invariance of $\mathcal{V}_{RS}$ show that it must be proportional to δ_{RS} i.e. $\mathcal{V}_{RS} = \mathcal{V}\,\delta_{RS}$. Now, using the completeness relation (2.5) we find:

$$\mathcal{V} = 1/2\,\delta^{RS}\mathcal{V}_{RS}$$

$$
\begin{aligned}
&= -1/2\, G^{AB} x^{\mu}_{,A} x^{\nu}_{,B} R_{\mu\rho\sigma\nu} \delta^{RS} n^{\rho}_{R} n^{\sigma}_{S} \\
&= 1/2\, G^{AB} x^{\mu}_{,A} x^{\nu}_{,B} \left(R_{\mu\nu} + G^{CD} x^{\rho}_{,C} x^{\sigma}_{,D} R_{\mu\rho\sigma\nu} \right). \qquad (4.49)
\end{aligned}
$$

Making use of a representation of the Ricci tensor $R_{\mu\nu}$ in terms of the Kinnersley null tetrad, namely:

$$
R_{\mu\nu} = \frac{2Q^2}{\Sigma^2} \left(m_{(\mu} \bar{m}_{\nu)} + (\Delta/2\Sigma) l_{+(\mu} l_{-\nu)} \right), \qquad (4.50)
$$

we are able to calculate the first term of equation (6.7) as follows:

$$
G^{AB} x^{\mu}_{,A} x^{\nu}_{,B} R_{\mu\nu} = \pm 2(\mp R_{\mu\nu} l^{\mu}_{\pm} \xi^{\nu}) = \frac{2Q^2}{\Sigma^2} l^{\mu}_{\pm} \xi_{\mu} = -\frac{2Q^2}{\Sigma^2}. \qquad (4.51)
$$

To calculate the second term of equation (6.7) we use the Gauss-Codazzi equations [46] for a 2-surface S embedded in a 4-dimensional spacetime. Namely:

$$
R^{(2)}_{ABCD} = \left(K_{RAC} K^{R}{}_{BD} - K_{RAD} K^{R}{}_{BC} \right) + R_{\mu\rho\sigma\nu} x^{\mu}_{,A} x^{\rho}_{,B} x^{\sigma}_{,C} x^{\nu}_{,D}. \qquad (4.52)
$$

Contracting (4.52) over A and C, and then B and D, one finds that the scalar curvature on S is just the sectional curvature in the tangent plane of S i.e.:

$$
R^{(2)} = G^{AC} G^{BD} R_{\mu\rho\sigma\nu} x^{\mu}_{,A} x^{\rho}_{,B} x^{\sigma}_{,C} x^{\nu}_{,D} \qquad (4.53)
$$

which is identically the second term in equation (4.49) up to the sign. Finally:

$$
\begin{aligned}
\mathcal{V} &= -\frac{1}{2} \left(R^{(2)} + 2\frac{Q^2}{\rho^4} \right) \\
&= 2 \left(\frac{Q^2(r^2 - a^2\cos^2\theta) - Mr(r^2 - 3a^2\cos^2\theta)}{\Sigma^3} \right), \qquad (4.54)
\end{aligned}
$$

where we have used the fact that $R^{(2)} = -F''$.

It remains to determine the normal fundamental form μ_{RSA}. Now as $\mu_{RSA} = \mu_{[RS]A}$, we can write $\mu_{RSA} = \mu_A \epsilon_{RS}$. It is then straightforward to verify that under the gauge transformation (4.40) μ_{RSA} transforms as:

$$
\mu_{RSA} \mapsto \tilde{\mu}_{RSA} = \mu_{RSA} + \epsilon_{RS} x^{\mu}_{,A} \partial_{\mu} \Psi, \qquad (4.55)
$$

or in light of the previous definition:

$$
\mu_A \mapsto \tilde{\mu}_A = \mu_A + x^{\mu}_{,A} \partial_{\mu} \Psi. \qquad (4.56)
$$

We define n_R over $S_\pm$ by parallel transport along a principal null trajectory and then by Lie transport along trajectories of the Killing vector, effectively fixing a gauge. That is on $S_\pm$:

$$l^\mu_\pm n^\nu_{R\ ;\mu} = 0, \qquad \xi^\mu_\pm n^\nu_{R\ ;\mu} = n^\mu_R \xi_{\nu;\mu}. \tag{4.57}$$

With this covariantly constant definition of n_R, using equation (4.41) in Appendix B, we find that:

$$\begin{aligned} \mu_{RS1} &= n^\mu_R l^\nu_\pm n_{S\mu;\nu} = 0, \\ \mu_{RS0} &= n^\mu_R n^\nu_S \xi_{\mu;\nu} = 1/2\ \epsilon_{RS}(n^\mu_2 n^\nu_3 - n^\mu_3 n^\nu_2)\xi_{\mu;\nu} = i\epsilon_{RS} M^\mu \bar{M}^\nu \xi_{\mu;\nu}. \end{aligned} \tag{4.58}$$

In order to take advantage of the decomposition of $\xi_{\mu;\nu}$ in terms of the Kinnersley null tetrad (4.4), we note that $M_\pm$ and m are related by the following null rotation:

$$M_\pm = m + E l_\pm, \tag{4.59}$$

where $E = \xi \cdot m$. Thus:

$$\mu_{RS0} = -\mu\ \epsilon_{RS}, \tag{4.60}$$

where $\mu = -a(1-F)\cos\theta/\Sigma$. If we let $\ell_{\pm A} = x^\mu_{,A} l_{\pm\mu}$ then we can write the normal fundamental form in this gauge as:

$$\mu_{RSA} = \mu \ell_{\pm A} \epsilon_{RS}, \tag{4.61}$$

so that here $\mu_A = \mu\ \ell_{\pm A}$.

However, a more convenient choice of gauge has $\mu_{RSA} \propto \epsilon_{RS}\eta_A$ where $\eta_A = x^\mu_{,A}\xi_\mu$ is a Killing vector on $S_\pm$. This can be seen explicitly:

$$\begin{aligned} \nabla_A \eta_B &= x^\mu_{,A} \nabla_\mu \left(\xi_\nu x^\nu_{,B}\right), \\ &= \xi_\nu x^\mu_{,A} \nabla_\mu x^\nu_{,B} + x^\nu_{,B} x^\mu_{,A} \nabla_\mu \xi_\nu. \end{aligned} \tag{4.62}$$

The antisymmetry of first term in A and B follows because as $x^\mu_{,A}$ are coordinate vector fields (see section 2.1). The antisymmetry of the second term follows from Killing's equations for the four-dimensional Killing field ξ.

This corresponds to a choice of the function Ψ on S such that $\eta_A \propto \tilde{\mu}_A = \mu\, \ell_{\pm A} + x^{\mu}_{,A}\partial_\mu \Psi$. If we let $\Psi = \Psi(r)$, then it follows that on S :

$$x^{\mu}_{,A}\partial_\mu \Psi = \mp \Psi'(F\ell_{\pm A} - \eta_A). \tag{4.63}$$

Clearly, if $\Psi' = \pm\mu/F$, then $\tilde{\mu}_A = (\mu/F)\eta_A$. With this choice of gauge we find that the equations of motion reduce to:

$$\left(\Box + \mathcal{V} + \mu^2/F\right)\tilde{\Phi}_R + 2\frac{\mu}{F}\epsilon_{RS}\eta^A\partial_A\tilde{\Phi}^S = 0, \tag{4.64}$$

where:

$$\mu = -\frac{a(1-F)\cos\theta}{\Sigma}, \tag{4.65}$$

$$\mathcal{V} = 2\left(\frac{Q^2(r^2 - a^2\cos^2\theta) - Mr(r^2 - 3a^2\cos^2\theta)}{\Sigma^3}\right). \tag{4.66}$$

Equation (4.64) can also be written in the form:

$$[G^{AB}(\delta_{RT}\nabla_A + \epsilon_{RT}\mathcal{A}_A)(\delta_{TS}\nabla_B + \epsilon_{TS}\mathcal{A}_B) + \delta_{RS}\mathcal{V}]\tilde{\Phi}^S = 0, \tag{4.67}$$

where $\mathcal{A}_A \equiv \mu\eta_A/F = (-\mu, \pm\mu/F)$ and we used the identity $G^{AB}\nabla_A(\mu\eta_B/F) = 0$. Here $\mathcal{A}_A$ plays the role of a vector potential while $\mathcal{V}$ is the scalar potential. Notice that the time component of $\mathcal{A}_A$, as well as $\mathcal{V}$, are finite everywhere, while the space component of $\mathcal{A}_A$ diverges at the static limit surface. But this divergence can be removed by a simple world-sheet coordinate transformation:

$$d\tilde{t} = du_{\pm} \mp F^{-1}(r)dr, \qquad d\tilde{r} = dr. \tag{4.68}$$

The perturbation equation still takes the form (4.67) but now the potentials are given by:

$$\tilde{\mathcal{A}}_A = (-\mu, 0), \qquad \tilde{\mathcal{V}} = \mathcal{V}, \tag{4.69}$$

that is, the potentials $(\tilde{\mathcal{A}}_A,\ \mathcal{V})$ are finite everywhere. There is however a divergence at the static limit surface in the time component of $\tilde{\mathcal{A}}^A$, but such situations are well-known from ordinary electromagnetism; this divergence does not destroy the regularity of the solution.

We conclude by noting that if $\mu = 0$, either in the equatorial plane of the Kerr-Newman spacetime ($\theta = \pi/2$) or in the non-rotating case ($a = 0$) the stringon propagation equations are separable and take the simple form:

$$\Box\Phi^R = \mathcal{V}\Phi^R. \tag{4.70}$$

4.6 String-Hole Physics

To conclude this chapter we discuss some problems connected with the proposed string-hole model of two-dimensional black and white holes. The central observation made in this chapter is that the interaction of a cosmic string with a 4-D black hole in which the string is trapped by the 4-D black hole opens new channels for the interaction of the black hole with the surrounding matter. The corresponding new degrees of freedom are related to excitations of the cosmic string (stringons). These degrees of freedom can be identified with physical fields propagating in the geometry of the 2-D string hole. There are two types of string holes corresponding to two types of the principal Killing surfaces S_+ and S_-. The first of them has the geometry of a 2-D black hole while the second has the geometry of a 2-D white hole. The physical properties of 'black' and 'white' string holes are different. For a regular initial state a 'black' string hole at late time is a source of a steady flux of thermal 'stringons'. This effect is an analog of the Hawking radiation [47]. In the simplest case when a stationary cosmic string is trapped by a Schwarzschild black hole, so that the string hole has 2-D Schwarzschild metric, the Hawking radiation of stringons was investigated in [48]. For such string holes their event horizon coincides with the event horizon of the 4-D black hole, and the temperature of the 'stringon' radiation coincides with the Hawking temperature of the 4-D black hole. For this reason the thermal excitations of the cosmic string will be in the state of thermal equilibrium with the thermal radiation of the 4-D black hole.

The situation is different in the general case when a stationary string is trapped by a rotating charged black hole. For the Kerr-Newman black hole the static limit surface is located outside the event horizon. The event horizon of the 2-D string hole does not coincide with the Kerr-Newman black hole horizon, except for the case where the cosmic string goes along the symmetry axis .

Notice that the (outer) horizon of the 2-D black hole coincides with the static limit of the 4-D rotating black hole. The 2-D surface gravity, which is proportional

to the 2-D temperature, is given by:

$$\kappa^{(2)} = \frac{1}{2}\frac{dF}{dr}\bigg|_{r=r_{+}} = \frac{\sqrt{M^2 - Q^2 - a^2\cos^2\theta}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - a^2\cos^2\theta}}. \tag{4.71}$$

The surface gravity of the 4-D Kerr-Newman black hole is:

$$\kappa^{(4)} = \frac{\sqrt{M^2 - Q^2 - a^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - a^2}}, \tag{4.72}$$

and then it can be easily shown that:

$$\kappa^{(2)} \geq \kappa^{(4)}. \tag{4.73}$$

That is to say, the 2-D temperature is higher than the 4-D temperature (except at the poles where they coincide) and it is always positive. Even if the 4-D black hole is extreme, the 2-D temperature is non-zero.

The reason why the temperature of a 2-D black hole differs from the temperature of the 4-dimensional Kerr-Newman black hole can be qualitatively explained if we note that for quanta located on the string surface (stringons) the angular momentum and energy are related. In the geometric optics approximation, a massless stringon propagating outwards (towards spatial infinity) on the world-sheet follows a null, geodesic trajectory on the world-sheet ([49]). While this trajectory is geodesic with respect to the geometry of the world-sheet, it is not geodesic in the background spacetime. As the stringon propagates outward, it's angular momentum and energy are not conserved due to the action of the string tension on the stringon. This alters the temperature of quanta observed at infinity.

In the general case ($a \neq 0$), a principal Killing surface in the Kerr-Newman spacetime is not geodesic. This property might have some interesting physical applications. Consider a black string hole and choose a point p inside its event horizon but outside the event horizon of the 4-dimensional Kerr-Newman black hole. Consider a timelike line γ_0 representing a static observer located outside the horizon of the 2-D black hole at $r = r_0$. There evidently exists an ingoing principal null ray crossing γ_0 and passing through p. It was shown in section 4.4 that there exists a future-directed 4-D null geodesic which begins at p and crosses γ_0. In other words, a causal signal

from p propagating in the 4-D embedding spacetime can connect points of the 2-D string hole interior with its exterior. For this reason stringons propagating inside the 2-D string hole can interact with the stringons in the 2-D string hole exterior. Such an interaction from the 2-D point of view is acausal. This interaction of Hawking stringons with their quantum correlated partners, created inside the string hole horizon might change the spectrum of the Hawking radiation, as well as its higher correlation functions.

4.7 Appendix: String Black Holes and Dilaton-Gravity

In this appendix, we show that the 2-D string holes, can also be obtained as solutions of 2-D dilaton gravity with a suitably chosen dilaton potential. To be more specific, we consider the following action of 2-D dilaton-gravity (see Louis-Martinez and Kunstatter [42]):

$$\mathcal{S} = \frac{1}{2\pi}\int dtdx\,\sqrt{-g}\,e^{-2\phi}\,[R + 2(\nabla\phi)^2 + V(\phi)], \tag{4.74}$$

where the dilaton potential $V(\phi)$ will be specified later. In 2-dimensions we can choose the conformal gauge:

$$g_{\mu\nu} = e^{2\rho} \times \text{diag.}(-1,\,1), \qquad \rho = \rho(t,x), \tag{4.75}$$

so that:

$$R = 2e^{-2\rho}(\rho_{,tt} - \rho_{,xx}). \tag{4.76}$$

The action (4.74) then takes the form:

$$\mathcal{S} = \frac{1}{\pi}\int dtdx\,e^{-2\phi}\,[\rho_{,tt} - \rho_{,xx} + \phi_{,x}^2 - \phi_{,t}^2 + \frac{1}{2}e^{2\rho}V(\phi)]. \tag{4.77}$$

The corresponding field equations read:

$$\begin{aligned} \rho_{,xx} - \rho_{,tt} + \phi_{,tt} - \phi_{,xx} + \phi_{,x}^2 - \phi_{,t}^2 + \tfrac{1}{4}e^{2\rho}(V' - 2V) = 0, \\ \phi_{,xx} - \phi_{,tt} + 2(\phi_{,t}^2 - \phi_{,x}^2) + \tfrac{1}{2}e^{2\rho}V = 0, \end{aligned} \tag{4.78}$$

where $V' \equiv dV/d\phi$. Now consider the special solutions:

$$\rho = \rho(x), \qquad \phi = \phi(x) \tag{4.79}$$

and introduce the coordinate r :

$$\frac{dr}{F(r)} = dx, \qquad e^{2\rho} = F. \tag{4.80}$$

Then the metric (4.75) leads to:

$$dS^2 = -F(r)dt^2 + F^{-1}(r)dr^2, \tag{4.81}$$

which is the form of our 2-D string holes (4.29), in the coordinates defined by:

$$d\tilde{t} = du_{\pm} \mp F^{-1}(r)dr, \qquad d\tilde{r} = dr. \tag{4.82}$$

It still needs to be shown that (4.79)-(4.80) is actually a solution to equations (4.78). The equations reduce to:

$$\begin{aligned} \phi_{,rr} - \phi_{,r}^2 + \tfrac{F_{,r}}{F}\phi_{,r} - \tfrac{1}{4F}(V' - 2V) &= \tfrac{F_{,rr}}{2F}, \\ \phi_{,rr} + \tfrac{F_{,r}}{F}\phi_{,r} - 2\phi_{,r}^2 + \tfrac{1}{2F}V &= 0. \end{aligned} \tag{4.83}$$

It can now be easily verified that both equations are solved by a "logarithmic dilaton" provided the dilaton potential takes the form:

$$V(\phi) = [\frac{2}{r^2}(rF)_{,r}]_{|r=e^{-\phi}/\lambda}, \tag{4.84}$$

$$\phi = -\log(\lambda r), \qquad \lambda = \text{const.} \tag{4.85}$$

for an arbitrary function $F(r)$. For our 2-D string holes, $F(r)$, is given by equation (3.32). The dilaton potential (4.84) then takes the explicit form:

$$V(\phi) = 2\lambda^2 e^{2\phi}[1 - \frac{4Me^{-\phi}/\lambda - Q^2}{e^{-2\phi}/\lambda^2 + a^2 - ab} + \frac{2e^{-\phi}(2Me^{-2\phi}/\lambda^2 - Q^2e^{-\phi}/\lambda)}{\lambda(e^{-2\phi}/\lambda^2 + a^2 - ab)^2}]. \tag{4.86}$$

This result holds for the general cone strings. A somewhat simpler expression is obtained for strings in the equatorial plane:

$$V(\phi) = 2\lambda^2 e^{2\phi}[1 - Q^2\lambda^2 e^{2\phi}]; \qquad \theta = \pi/2 \tag{4.87}$$

Chapter 5

Principal Weyl Surfaces

In this chapter we introduce a method to describe families of timelike two-surfaces using the spin coefficients of an associated complex null tetrad. The two-surfaces are integral submanifolds of the two real null vectors of the tetrad. This was suggested by Geroch, Held and Penrose [50] (GNP) using their modification of the Newman-Penrose (NP) formalism [51]. Here, we will use the more commonly seen Newman-Penrose description. While the GHP formalism is better suited for the study of two-surfaces, the NP formalism is more common in the literature of exact solutions.

We will find that minimal timelike two-surfaces are described simply in this formalism. A minimal surface is associated with a tetrad for which certain spin coefficients vanish. We will use this fact to study distinguished minimal surfaces in certain algebraically special spacetimes. This will allow us to generalize the Principal Killing surfaces of the previous chapter.

Principal Killing surfaces were studied in detail in $2+1$-dimensional gravity by Frolov, Hendy and Larsen [52]. In $2+1$ dimensions, a Principal Killing surface, defined as the surface formed by a Killing vector ξ and an eigenvector l of the anti-symmetric tensor $\xi_{\mu;\nu}$, was minimal if and only if the eigenvector l is geodesic. This is also true in $3+1$ dimensions. However in $2+1$ dimensions this condition places a relatively simple constraint on the metric. In $3+1$ dimensions the situation is considerably more complicated and such a condition has not been found.

Here, we will examine algebraically special $3+1$ dimensional spacetimes (see page 63), namely the vacuum Kerr-Schild spacetimes [17] and the vacuum Petrov type-D spacetimes (the Kerr spacetime is a member of both families). We will generalize principal Killing surfaces by examining what we call Principal Weyl surfaces; these minimal two-surfaces are tangent to a shear-free geodesic vector. We begin

this chapter by indicating how one can describe two-surfaces using a NP tetrad. We then introduce the Principal Weyl surfaces and then we will examine these surfaces in Kerr-Schild spacetimes. We work with a metric of signature $(1,-1,-1,-1)$ in this chapter to make contact with the literature on algebraically spacetimes.

5.1 Classification of timelike two-surfaces embedded in curved spacetime

Our goal is to study timelike two-surfaces S with spacetime embedding $x^\mu = x^\mu(\zeta^A)$ where ζ^A, $A = 0, 1$, are coordinates on S. The two tangent vectors

$$x^\mu_{,A} = \frac{\partial x^\mu}{\partial \zeta^A} \tag{5.1}$$

formed a coordinate basis for tangent vectors to S at each point on S (recall page 20). However, it is convenient here to work with an orthonormal basis of tangent vectors $\{E^\mu_A\}$:

$$g_{\mu\nu} E^\mu_A E^\nu_B = \eta_{AB}. \tag{5.2}$$

As the surface S is timelike everywhere we can always choose one vector field E_0 to be timelike. We can also introduce a pair of unit normals to the surface, n^μ_R $(R = 2, 3)$, defined up to local rotations by

$$g_{\mu\nu} n^\mu_R n^\nu_S = -\delta_{RS}, \quad g_{\mu\nu} E^\mu_A n^\nu_R = 0. \tag{5.3}$$

We thus have an orthonormal tetrad $\{E_A, n_R\}$ at each point on the surface S. The second fundamental form of the two-surface S is given by

$$K_{R,AB} = g_{\mu\nu} n^\mu_R \nabla_A E^\nu_B, \tag{5.4}$$

where $\nabla_A = E^\mu_A \nabla_\mu$ is the projection onto S of the spacetime covariant derivative.

The orthonormal tetrad is related to a complex null tetrad as follows:

$$k^\mu = 1/\sqrt{2}(E^\mu_0 - E^\mu_1), \quad l^\mu = 1/\sqrt{2}(E^\mu_0 + E^\mu_1), \quad m^\mu = 1/\sqrt{2}(n^\mu_2 + i n^\mu_3). \tag{5.5}$$

This complex null tetrad $\{k, l, m, \bar{m}\}$ satisfies the relations

$$k^2 = l^2 = m^2 = 0, \quad k \cdot l = 1, \quad k \cdot m = l \cdot m = 0, \quad m \cdot \bar{m} = -1. \tag{5.6}$$

Note that up to normalization the two null tangent vectors k and l are independent of our choice of orthonormal tetrad. At each point on the timelike two-surface there are only two null tangent directions. Thus a given timelike two-surface is associated with a complex null tetrad $\{k, l, m, \bar{m}\}$ which is unique up to boosts and local rotations.

Conversely, we recall Frobenius' theorem (see Wald [18] or chapter 2) for two-dimensional submanifolds: a necessary and sufficient condition that the two distinct vector fields, k and l, possess a family of integral submanifolds is that

$$[k, l] = \alpha k + \beta l \tag{5.7}$$

where $\alpha = \alpha(x^\mu)$ and $\beta = \beta(x^\mu)$ are functions of the spacetime coordinates x^μ. Such vector fields k and l are said to be surface-forming. Further, if $[k, l] = 0$ then the two tangent vectors form a coordinate basis on each submanifold. Thus we can define a family of two-surfaces by specifying two commuting vector fields in a spacetime.

Consider a complex null tetrad $\{k, l, m, \bar{m}\}$ defined everywhere in spacetime. One can compute the commutator $[k, l]$ in this tetrad in terms of the connection coefficients

$$[k, l] = -(\gamma + \bar{\gamma})k - (\epsilon + \bar{\epsilon})l + (\bar{\tau} + \pi)m + (\tau + \bar{\pi})\bar{m}. \tag{5.8}$$

Thus we see that k and l are surface-forming if and only if $\tau + \bar{\pi} = 0$ [53].

Suppose then that $\tau + \bar{\pi} = 0$. In this case k and l possess a family of timelike two-dimensional integral surfaces. We introduce the normalization $\tilde{k} = A(\mathbf{x})k$ and $\tilde{l} = B(\mathbf{x})l$ so that

$$\left[\tilde{k}, \tilde{l}\right] = 0. \tag{5.9}$$

In this case we can find coordinates $X^A = (u, v)$ on each surface such that

$$\tilde{k}^\mu = \frac{\partial x^\mu}{\partial u}, \quad \tilde{l}^\mu = \frac{\partial x^\mu}{\partial v}. \tag{5.10}$$

In general, it is not possible to retain the normalization $\tilde{k} \cdot \tilde{l} = 1$ while satisfying (5.9). The induced metric on the two-surface with coordinates $X^A = (u, v)$ is then

$$dS^2 = G_{AB} dX^A dX^B = 2AB du dv. \tag{5.11}$$

We are particularly interested in minimal surfaces (which are solutions of the Nambu-Goto equations of motion). A surface is minimal if and only if

$$G^{AB}K_{RAB} = K_R{}^A{}_A = 0. \tag{5.12}$$

In particular, the complex quantity $K_2{}^A{}_A + iK_3{}^A{}_A$ vanishes if and only if the surface in question is minimal. We can compute the quantity $K_2{}^A{}_A + iK_3{}^A{}_A$ in the metric (5.11) using the complex null tetrad $\{k, l, m, \bar{m}\}$. We find that

$$\begin{aligned} K_2{}^A{}_A + iK_3{}^A{}_A &= -\sqrt{2}\, m^\mu (Dl_\mu + \Delta k_\mu), \\ &= \sqrt{2}\,(\bar{\pi} - \tau). \end{aligned} \tag{5.13}$$

Thus, given that the vector fields k and l are surface-forming so that $\bar{\pi} + \tau = 0$, the corresponding integral two-surfaces are minimal if and only if $\bar{\pi} - \tau = 0$ or, equivalently, if and only if $\pi = \tau = 0$. In the next section we will see that this property enables us to find a special class of minimal surfaces in algebraically special spacetimes.

Suppose that the vector fields k and l are surface-forming so that $\tau + \bar{\pi} = 0$. The integral two-surfaces of these vector fields are stationary if there exists scalar functions $a(x^\mu)$ and $b(x^\mu)$ such that the linear combination

$$\xi = ak + bl \tag{5.14}$$

is a Killing vector. The vector ξ is a Killing vector if and only if $\xi_{(\mu;\nu)} = 0$, or in terms of a and b:

$$\begin{aligned} Db &= (\epsilon + \bar{\epsilon})b, \\ \Delta a &= -(\gamma + \bar{\gamma})a, \\ \sigma a &= \bar{\lambda} b, \\ \Delta b + Da &= (\gamma + \bar{\gamma})b - (\epsilon + \bar{\epsilon})a, \\ \delta b &= (\bar{\alpha} + \beta)b - \kappa a + \bar{\pi} b, \\ \delta a &= -(\bar{\alpha} + \beta)a + \bar{\nu} b - \tau a, \\ (\rho + \bar{\rho})a &= (\mu + \bar{\mu})b, \end{aligned} \tag{5.15}$$

where we denote the directional derivatives associated with this tetrad (see Appendix A) as

$$D = k^\mu \nabla_\mu, \;\; \Delta = l^\mu \nabla_\mu, \;\; \delta = m^\mu \nabla_\mu, \;\; \bar{\delta} = \bar{m}^\mu \nabla_\mu. \tag{5.16}$$

5.2 Principal Weyl surfaces

In this section we consider minimal surfaces in spacetimes possessing a geodesic shear-free null congruence. In algebraically special vacuum spacetimes, the Weyl tensor possesses repeated principal null directions (repeated null eigenvectors) which are geodesic and shear-free. [1] We will find that under certain conditions such spacetimes contain special families of minimal surfaces corresponding to each principal null direction.

Let $\{k, l, m, \bar{m}\}$ be any complex null tetrad. We can then perform the following null rotation about k:

$$\begin{aligned} \hat{l} &= l + \bar{E}m + E\bar{m} + E\bar{E}k, \\ \hat{m} &= m + Ek, \\ \hat{k} &= k. \end{aligned} \tag{5.17}$$

One can verify then that in this new tetrad $\hat{\kappa} = \kappa$, $\hat{\sigma} = \sigma + E\kappa$ and $\hat{\rho} = \rho + \bar{E}\kappa$. Further τ and π transform as (see Appendix A)

$$\begin{aligned} \hat{\tau} &= \tau + \bar{E}\sigma + E\rho + E\bar{E}\kappa, \\ \hat{\pi} &= \pi + 2\bar{E}\epsilon + \bar{E}^2\kappa + D\bar{E}, \end{aligned} \tag{5.18}$$

Now consider an algebraically special spacetime where the vector field k is geodesic (so $\kappa = 0$), shear-free (so that $\sigma = 0$) and diverging (so that $\rho \neq 0$). Setting $E = -\tau/\rho$ it is easy to see that $\hat{\tau} = 0$ because $\kappa = \sigma = 0$. Further, using the appropriate Ricci

[1] Recall the Goldberg-Sachs theorem for vacuum spacetimes: if the Weyl tensor possesses a repeated principal null direction then the corresponding null vector is geodesic and shear-free (page 63).

identities (Newman and Penrose [51]) for the tetrad $\{k, l, m, \bar{m}\}$ where $\kappa = \sigma = 0$ (noting that $\Psi_1 = 0$ in this tetrad), namely,

$$D\rho = \rho^2 + (\epsilon + \bar{\epsilon})\rho, \tag{5.19}$$

$$D\tau = (\tau + \bar{\pi})\rho + (\epsilon - \bar{\epsilon})\tau, \tag{5.20}$$

one can show that

$$D\bar{E} = -(\pi + 2\bar{E}\epsilon) \tag{5.21}$$

so that $\hat{\pi} = 0$. Similarly one can verify that $\hat{\lambda} = 0$. It is also straightforward to show that the null vector $\hat{l}$ is independent of the choice of the null vector l.

To summarise, the tetrad $\left\{\hat{k}, \hat{l}, \hat{m}, \hat{\bar{m}}\right\}$ was obtained from the tetrad $\{k, l, m, \bar{m}\}$ by a null rotation (5.17) with $E = -\tau/\rho$. Further, if $\kappa = \sigma = 0$ the following spin coefficients vanish in the transformed tetrad:

$$\hat{\kappa} = \hat{\sigma} = \hat{\tau} = \hat{\pi} = \hat{\lambda} = 0. \tag{5.22}$$

In fact, up to boosts and rotations of the vector fields m and $\bar{m}$, this transformed tetrad is the only complex null tetrad containing the vector field k that satisfies (5.22). One can verify (see Appendix A) that the vanishing of the connection coefficients (5.22) is unchanged by such transformations (5.42).

Using the results of the previous section, we see that not only are the null vector fields $\hat{k}$ and $\hat{l}$ surface-forming, but that their integral submanifolds are minimal. We call this family of two-surfaces principal Weyl surfaces, because the vector $\hat{k}$ is a principal null vector of the Weyl tensor in a vacuum spacetime. Furthermore, these principal Weyl surfaces are uniquely associated with the shear-free null geodesic vector field k.

We now wish to consider when these principal Weyl surfaces are stationary, i.e. under what circumstances a Killing vector ξ is tangent to the principal Weyl surfaces. We consider a stationary algebraically special vacuum spacetime, which possesses at least one Killing vector field ξ, and a diverging geodesic shear-free null congruence with tangent vector k. Thus we can find a null tetrad $\left\{k = \hat{k}, \hat{l}, \hat{m}, \hat{\bar{m}}\right\}$ where $\hat{\kappa} = \hat{\sigma} = \hat{\tau} = \hat{\pi} = 0$. If the Killing vector ξ is tangent to the corresponding

principal Weyl surfaces (so that $\xi = a\hat{k} + b\hat{l}$), then we find

$$\begin{aligned} \hat{k}^\nu \nabla_\nu \xi_{(t)\,\mu} &= \hat{D}\xi_{(t)\,\mu}, \\ &= (\hat{D}a + (\hat{\epsilon} + \bar{\hat{\epsilon}})a)\hat{k}_\mu. \end{aligned} \tag{5.23}$$

Thus, $\hat{k}$ is an eigenvector of the antisymmetric tensor $\nabla_\nu \xi_\mu$.

The converse is not quite true, since an eigenvector $\tilde{k}$ of $\nabla_\nu \xi_\mu$, and the corresponding Killing vector ξ, are not necessarily surface-forming. Specifically, in a tetrad $\left\{\tilde{k}, \tilde{l}, \tilde{m}, \bar{\tilde{m}}\right\}$ where $\xi = a\tilde{k} + b\tilde{l}$ we have:

$$\mathcal{L}_\xi \tilde{k}_\mu = ((\tilde{\gamma} + \bar{\tilde{\gamma}})b - \tilde{D}a)\tilde{k}_\mu + ((\tilde{\epsilon} + \bar{\tilde{\epsilon}})b - \tilde{D}b)\tilde{l}_\mu - (\tilde{\pi} + \bar{\tilde{\tau}})b\tilde{m}_\mu - (\bar{\tilde{\pi}} + \tilde{\tau})b\bar{\tilde{m}}_\mu. \tag{5.24}$$

Note that the coefficient of $\tilde{l}$ vanishes because of Killing's equations (5.15) for ξ. Thus ξ and $\tilde{k}$ are surface-forming provided $\tilde{\pi} + \bar{\tilde{\tau}} = 0$ (we do not consider the case where $b = 0$).

If we assume that ξ and $\tilde{k}$ are surface-forming, then because $\tilde{k}$ is an eigenvector of $\nabla_\nu \xi_\mu$, one can show that $\tilde{\pi}b - \bar{\tilde{\kappa}}a = 0$. We are interested in the situation where $\tilde{k}$ is geodesic and shear-free. In this case we find $\tilde{\pi} = \tilde{\tau} = 0$; in other words, the integral surfaces of $\tilde{k}$ and ξ are precisely the principal Weyl surfaces.

Our result is as follows: in a algebraically special spacetime with a geodesic, shearfree vector field k and a Killing vector ξ, which are surface-forming, the principal Weyl surfaces with a tangent vector k are stationary with respect to ξ, if and only if k is an eigenvector of $\nabla_\nu \xi_\mu$. This result holds in any algebraically special stationary spacetimes which contain principal Weyl surfaces.

Petrov type-D vacuum spacetimes were classified by Kinnersley [54]. They all possess two distinct geodesic shear-free congruences; the corresponding tangent vectors are each twice-repeated principal null directions of the Weyl tensor. The Kerr metric is a vacuum Petrov type-D metric (see section 4.1). All the Petrov type-D vacuum spacetimes are stationary and axisymmetric; they possess two Killing vectors which we denote $\xi_{(t)}$ and $\xi_{(\phi)}$. Kinnersley integrates the Newman-Penrose equations using a tetrad that contains the two principal null directions: $\{k, l, m, \bar{m}\}$, where k and l are the principal null directions of the Weyl tensor. There are several cases:

1. $\rho \neq 0$: these spacetimes contain two geodesic, shear-free and diverging null congruences. Thus, the null rotation (5.17) is well-defined. These families of spacetimes are labelled type I, type II and type III. Type I metrics include the three NUT metrics [55], type II includes the Kerr spacetime and type III include the C-metric [56]. Let $\hat{l}$ be the null vector defined by the rotation

$$\hat{l} = l + E\bar{m} + \bar{E}m + E\bar{E}k \tag{5.25}$$

where $E = -\tau/\rho$. The principal Weyl surfaces are then the integral two-surfaces of the two null vector fields $\hat{k} = k$ and $\hat{l}$. It is then straightforward to check whether $\hat{m} \cdot \xi_{(t)} = (m + Ek) \cdot \xi_{(t)}$ vanishes in the Kinnersley tetrad [2]. If this quantity vanishes then the principal Weyl surface is stationary since $\xi_{(t)}$ is tangent to the world sheet. One can verify that the principal Weyl surfaces are stationary in all type I metrics and in all type II metrics but not in the type III metrics. Thus in the type I and II metrics, the Killing vector is given by $\xi_{(t)} = a\hat{k} + b\hat{l}$ where $\hat{k} = k$ is a principal null vector of the Weyl tensor and l is given by the null rotation (5.25). We conclude, in these two cases, that k is an eigenvector of the antisymmetric tensor $\nabla_\nu \xi_{(t)\,\mu}$.

2. $\rho = 0$: these spacetimes contain two geodesic, shear-free and divergenceless null congruences. We have not considered such spacetimes here and we note that the null rotation (5.17) with $E = -\tau/\rho$ is ill-defined. While it is possible to show that π can be set to zero by a null rotation of the form (5.17), τ cannot be set to zero in a tetrad containing the geodesic, shear-free and diverging null vector field [57]. Thus we cannot find principal Weyl surfaces. These metrics are labelled the type IV metrics by Kinnersley [54].

[2] This was done using the Mathtensor package for Mathematica

5.3 Stationary Surfaces in Kerr-Schild spacetimes

In this section we will look at Principal Weyl surfaces in generalized Kerr-Schild spacetimes. A generalized Kerr-Schild spacetime admits a metric of the form

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + k_\mu k_\nu, \tag{5.26}$$

$$\hat{g}^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu \tag{5.27}$$

where k^μ is null, geodesic and shear-free with the metric g. The vector field k^μ is then geodesic and shear-free with respect to the metric $\hat{g}$ also (see Appendix B or [58]). When the resulting spacetime $\hat{V}$ is a vacuum spacetime then k is a repeated principal null vector of the Weyl tensor so $\hat{V}$ is algebraically special (by the Goldberg-Sachs theorem again, page 63). The Kerr-Newman spacetime is a Kerr-Schild spacetime [17].

In what follows we consider a generalized Kerr-Schild spacetime (5.26) where the vector field k is geodesic ($\hat{\kappa} = \kappa = 0$) [3], shear-free ($\hat{\sigma} = \sigma = 0$) and diverging ($\hat{\rho} = \rho \neq 0$). Thus we can always write down a complex null tetrad $\{k^\mu, l^\mu, m^\mu, \bar{m}^\mu\}$ with respect to the metric g such that $\pi = \tau = 0$ using the null rotation (5.17) with $E = -\tau/\rho$. The corresponding tetrad in $\hat{g}$ is $\{\hat{k}^\mu = k^\mu, \hat{l}^\mu = l^\mu - (1/2)k^\mu, \hat{m}^\mu = m^\mu, \hat{\bar{m}}^\mu = \bar{m}^\mu\}$; from Appendix B (page 97) we see that again $\hat{\pi} = \hat{\tau} = 0$.

Thus the principal Weyl surfaces associated with the vector field k in V are simply related to the principal Weyl surfaces associated with $\hat{k}$ in $\hat{V}$. If the principal Weyl surfaces in V are stationary, then we have

$$\xi = ak + bl \tag{5.28}$$

for some Killing vector ξ (in V) and some scalar fields a and b. Now $\hat{\xi}^\mu = \xi^\nu$ will not, in general, be a Killing vector in the spacetime $\hat{V}$. Nonetheless it is still tangent to the principal Weyl surfaces since

$$\hat{g}_{\mu\nu}\hat{m}^\mu\hat{\xi}^\nu = (g_{\mu\nu} + k_\mu k_\nu)m^\mu\xi^\nu = g_{\mu\nu}m^\mu\xi^\nu = 0. \tag{5.29}$$

[3] Quantities that appear with a hat ˆ have been evaluated in the metric $\hat{g}$; those without have been evaluated in g

It is interesting to ask under what conditions $\hat{\xi} = \xi$ will remain a Killing vector in $\hat{V}$.

We will examine the case where $g_{\mu\nu}$ is a flat metric, say $g_{\mu\nu} = \eta_{\mu\nu}$ where k is geodesic, shear-free and diverging ($\kappa = \sigma = 0, \rho \neq 0$). All the vacuum solutions of this type are known (Debney et. al [59]) and all possess at least one vector field $\hat{\xi}$ that is a Killing vector in both $\hat{V}$ and in flat spacetime ([57]). This family of solutions includes the Kerr spacetime.

Suppose that ξ^μ is a Killing vector in both the Kerr-Schild spacetime $\hat{V}$ and in flat spacetime. Then $\hat{\xi}^\mu = \xi^\mu$ satisifies Killing's equations in $\hat{V}$:

$$\hat{\mathcal{L}}_\xi \hat{g}_{\mu\nu} = \hat{\mathcal{L}}_\xi (\eta_{\mu\nu} + k_\mu k_\nu) = 0. \tag{5.30}$$

where $\hat{\mathcal{L}}_\xi$ is the Lie derivative with respect to the metric $\hat{g}$ in the direction of ξ. Now

$$\hat{\mathcal{L}}_\xi \eta_{\mu\nu} = \hat{\nabla}_\xi \eta_{\mu\nu} + \eta_{\mu\sigma} \hat{\nabla}_\nu \xi^\sigma + \eta_{\sigma\nu} \hat{\nabla}_\mu \xi^\sigma \tag{5.31}$$

where $\hat{\nabla}$ is the covariant derivative in $\hat{V}$. The first term on the righthand side of (5.31) $\hat{\nabla}_\xi \eta_{\mu\nu} = -\xi^\sigma (\eta_{\rho\nu} \hat{\Gamma}^\rho_{\mu\sigma} + \eta_{\rho\nu} \hat{\Gamma}^\rho_{\mu\sigma})$ where $\hat{\Gamma}^\rho_{\mu\sigma}$ are the Christoffel symbols for the metric on $\hat{V}$. Inserting this into (5.31) we find

$$\begin{aligned} \hat{\mathcal{L}}_\xi \eta_{\mu\nu} &= -\xi^\sigma (\eta_{\rho\nu} \hat{\Gamma}^\rho_{\mu\sigma} + \eta_{\rho\nu} \hat{\Gamma}^\rho_{\mu\sigma}) + \eta_{\mu\sigma} (\partial_\nu \xi^\sigma + \xi^\rho \hat{\Gamma}^\sigma_{\rho\nu}) + \eta_{\sigma\nu} (\partial_\mu \xi^\sigma + \xi^\rho \hat{\Gamma}^\sigma_{\mu\rho}) \\ &= \eta_{\mu\sigma} \partial_\nu \xi^\sigma + \eta_{\sigma\nu} \partial_\mu \xi^\sigma. \end{aligned} \tag{5.32}$$

If ξ is also a Killing vector in flat spacetime this term vanishes. In this case Killing's equations (5.30) reduce to

$$\hat{\mathcal{L}}_\xi \hat{g}_{\mu\nu} = 2 \hat{\mathcal{L}}_\xi (k_\mu k_\nu) = 0. \tag{5.33}$$

Working with a tetrad $\left\{\hat{k} = k, \hat{l}, \hat{m}, \bar{\hat{m}}\right\}$ with respect to the metric $\hat{g}$ in which $\hat{m}_\mu \xi^\mu = 0$ and contracting (5.33) with the appropriate combination of tetrad vectors one finds that $\hat{\mathcal{L}}_\xi k_\mu$ vanishes if Killing's equations are satisfied by ξ in $\hat{V}$:

$$\hat{\mathcal{L}}_\xi k_\mu = 0. \tag{5.34}$$

In particular, this means that ξ and k are surface-forming in $\hat{V}$, and that they are coordinate vector fields for some family of integral submanifolds embedded in $\hat{V}$. In

the tetrad where $\xi = ak + b\hat{l}$, we recall from the previous section (5.24) that $\hat{\mathcal{L}}_\xi k_\mu$ can be written as

$$\hat{\mathcal{L}}_\xi k_\mu = ((\hat{\gamma} + \hat{\bar{\gamma}})b - \hat{D}a)k_\mu + ((\hat{\epsilon} + \hat{\bar{\epsilon}})b - \hat{D}b)\hat{l}_\mu - (\hat{\bar{\tau}} + \hat{\pi})b\hat{m}_\mu - (\hat{\tau} + \hat{\bar{\pi}})b\hat{\bar{m}}_\mu. \quad (5.35)$$

where $\hat{D} = k^\mu \hat{\nabla}_\mu$. The coefficients of k and $\hat{l}$ can be set to zero by an appropriate normalization, and $\hat{\bar{\tau}} + \hat{\pi} = 0$ if k and $\hat{l}$ are surface-forming.

Summarising, if ξ is a Killing vector in flat spacetime then

$$\hat{\mathcal{L}}_\xi \hat{g}_{\mu\nu} = 2\hat{\mathcal{L}}_\xi(k_\mu k_\nu) \quad (5.36)$$

in the Kerr-Schild spacetime $\hat{V}$. Thus if ξ is a Killing vector in flat spacetime then ξ is a Killing vector in $\hat{V}$ if and only if $\hat{\mathcal{L}}_\xi k_\mu = 0$. In particular this is true for the tetrad associated with the principal Weyl surfaces in the Kerr-Schild spacetime since for this tetrad $\pi = \tau = 0$. Here, we recall that the vector k is an eigenvector of $\hat{\nabla}_{[\mu}\xi_{\nu]} = \hat{\nabla}_\mu \xi_\nu$ since for $\kappa = \pi = 0$

$$\hat{D}\xi_\mu = (\hat{D}a + (\hat{\epsilon} + \hat{\bar{\epsilon}})a)k_\mu. \quad (5.37)$$

Let S be the family of principal Weyl surfaces formed by k and l in flat spacetime. Then the family of surfaces S' formed by $\hat{k}$ and $\hat{l}$ in the Kerr-Schild spacetime with metric (5.26) is also a family of principal Weyl surfaces (minimal and with tangent vector k). Further if S is a stationary family of surfaces, so that the Killing vector ξ in flat spacetime is tangent to each surface, then the family S' is also stationary with Killing vector $\hat{\xi} = \xi$. Thus the properties of the principal Weyl surfaces in the Kerr-Schild spacetime can be understood by looking at the properties of the corresponding principal Weyl surfaces in flat spacetime.

We conclude by returning to the Kerr spacetime. As mentioned earlier the Kerr metric is a Kerr-Schild metric. In fact,the metric can be written in Eddington-Finkelstein coordinates $(u_\pm, r, \theta, \tilde{\phi}_\pm)$(4.26) as

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + (1 - F)l_{\mu\,\pm} l_{\nu\,\pm} \quad (5.38)$$

where $F = -\xi^2_{(t)}$. The factor $(1 - F)$ can be removed by appropriate normalization, giving the metric exactly the form of (5.26). The vectors $l_\pm$ are the principal null

vectors of the Kerr-Newman spacetime (recall the definition (4.1) in chapter 4). In ingoing Eddington-Finkelstein coordinates (4.25) l_+ takes the form $l_+^\mu = (0,1,0,0)$; in outgoing Eddington-Finkelstein coordinates l_- takes the form $l_-^\mu = (0,-1,0,0)$.

The two commuting Killing vectors of the Kerr spacetime (page 42) are given by $\xi^\mu_{(t)} = \delta^\mu_u$ and $\xi^\mu_{(\phi)} = \delta^\mu_{\bar{\phi}}$ in the Eddington-Finkelstein coordinates. Both these vector fields are also Killing vectors in flat spacetime. We now observe that

$$\hat{g}_{\mu\nu} l_+^\mu \xi^\nu_{(t)} = -1, \quad \hat{g}_{\mu\nu} l_+^\mu \xi^\nu_{(\phi)} = a \sin^2\theta. \tag{5.39}$$

Note that $\hat{g}_{\mu\nu} l_+^\mu \xi^\nu_{(\phi)} = \eta_{\mu\nu} l_+^\mu \xi^\nu_{(\phi)}$ and that in flat spacetime ($a = 0$) we have $\eta_{\mu\nu} l_+^\mu \xi^\nu_{(\phi)} = 0$. Thus the Killing vector $\xi^\nu_{(\phi)}$, which is spacelike everywhere, is not tangent to any principal Weyl surfaces with tangent vector l_+ in flat spacetime; the surfaces formed by $\xi^\nu_{(\phi)}$ and l_+ are null in flat spacetime and thus escape the discussion in this chapter.

On the other hand, $\hat{g}_{\mu\nu} l_+^\mu \xi^\nu_{(t)} = l_+^\mu \xi^\nu_{(t)} - 1$. The surfaces formed by the vector fields $l_\pm$ and $\xi_{(t)}$ in flat spacetime are minimal and timelike. These are in fact stationary principal Weyl surfaces embedded in flat spacetime. In the Kerr spacetime, the corresponding principal Weyl surfaces must also be stationary; these are precisely the principal Killing surfaces of Chapter 4.

5.4 Appendix A: The Newman-Penrose Formalism

We provide a few details of the formalism used in Chapter 5. We work with a metric of signature $(1,-1,-1,-1)$ and a normalization that ensures the Newman-Penrose tetrad $\{l, n, m, \bar{m}\}$ satisfies the completeness relation $g^{\mu\nu} = l^{(\mu} n^{\nu)} - m^{(\mu} \bar{m}^{\nu)}$:

$$l^\mu n_\mu = 1 \qquad m^\mu \bar{m}_\mu = -1. \tag{5.40}$$

We denote the directional derivatives of the tetrad by

$$l^\mu \nabla_\mu = D, \quad n^\mu \nabla_\mu = \Delta, \quad m^\mu \nabla_\mu = \delta, \quad \bar{m}^\mu \nabla_\mu = \bar{\delta}. \tag{5.41}$$

The spin coefficients are defined by

$$\begin{aligned}
Dl &= (\epsilon+\bar{\epsilon})l - \bar{\kappa}m - \kappa\bar{m} \\
\Delta l &= (\gamma+\bar{\gamma})l - \bar{\tau}m - \tau\bar{m} \\
\delta l &= (\bar{\alpha}+\beta)l - \bar{\rho}m - \sigma\bar{m} \\
Dn &= -(\epsilon+\bar{\epsilon})n + \pi m + \bar{\pi}\bar{m} \\
\Delta n &= -(\gamma+\bar{\gamma})n + \nu m + \bar{\nu}\bar{m} \\
\delta n &= -(\bar{\alpha}+\beta)n + \mu m + \bar{\lambda}\bar{m} \\
Dm &= \bar{\pi}l - \kappa n + (\epsilon-\bar{\epsilon})m \\
\Delta m &= \bar{\nu}l - \tau n + (\gamma-\bar{\gamma})m \\
\delta m &= \bar{\lambda}l - \sigma n + (\beta-\bar{\alpha})m \\
\bar{\delta}m &= \bar{\mu}l - \rho n + (\alpha-\bar{\beta})m
\end{aligned}$$

We give here the behaviour of the Newman-Penrose scalars under Lorentz transformations. Under the transformation:

$$\tilde{l} = al, \qquad \tilde{n} = a^{-1}n, \qquad \tilde{m} = e^{2i\theta}m, \tag{5.42}$$

the Newman-Penrose scalars transform according to:

$$\begin{aligned}
&\tilde{\kappa} = a^2e^{2i\theta}\kappa, \quad \tilde{\pi} = e^{-2i\theta}\pi, \qquad \tilde{\epsilon} = a(\epsilon + D(\tfrac{1}{2}\ln a + i\theta)), \\
&\tilde{\tau} = e^{2i\theta}\tau, \qquad \tilde{\nu} = a^{-2}e^{-2i\theta}\nu, \quad \tilde{\gamma} = a^{-1}(\gamma + \Delta(\tfrac{1}{2}\ln a + i\theta)), \\
&\tilde{\sigma} = ae^{4i\theta}\sigma, \quad \tilde{\mu} = a^{-1}\mu, \qquad \tilde{\beta} = e^{2i\theta}(\beta + \delta(\tfrac{1}{2}\ln a + i\theta)), \\
&\tilde{\rho} = a\rho, \qquad \tilde{\lambda} = a^{-1}e^{-4i\theta}\lambda, \quad \tilde{\alpha} = e^{-2i\theta}(\alpha + \bar{\delta}(\tfrac{1}{2}\ln a + i\theta)).
\end{aligned} \tag{5.43}$$

Under a null rotation about l,

$$\hat{l} = l, \qquad \hat{m} = m + cl, \qquad \hat{n} = n + \bar{c}m + c\bar{m} + c\bar{c}l, \tag{5.44}$$

the Newman-Penrose scalars transform as

$$
\begin{aligned}
\hat{\kappa} &= \kappa, \\
\hat{\epsilon} &= \epsilon + \bar{c}\kappa, \\
\hat{\sigma} &= \sigma + c\kappa, \\
\hat{\rho} &= \rho + \bar{c}\kappa, \\
\hat{\tau} &= \tau + \bar{c}\sigma + c\rho + c\bar{c}\kappa, \\
\hat{\alpha} &= \alpha + \bar{c}\epsilon + \bar{c}\rho + \bar{c}^2\kappa, \\
\hat{\beta} &= \beta + \bar{c}\sigma + c\epsilon + c\bar{c}\kappa, \\
\hat{\pi} &= \pi + 2\bar{c}\epsilon + \bar{c}^2\kappa + D\bar{c}, \\
\hat{\gamma} &= \gamma + c\alpha + \bar{c}(\tau + \beta) + c\bar{c}(\rho + \epsilon) + \bar{c}^2\sigma + c\bar{c}^2\kappa, \\
\hat{\lambda} &= \lambda + \bar{c}\pi + 2\bar{c}\alpha + \bar{c}^2(\rho + 2\epsilon) + \bar{c}^3\kappa + \bar{c}D\bar{c} + \bar{\delta}\bar{c}, \\
\hat{\mu} &= \mu + 2\bar{c}\beta + c\pi + \bar{c}^2\sigma + 2c\bar{c}\epsilon + c\bar{c}^2\kappa + cD\bar{c} + \delta\bar{c}, \\
\hat{\nu} &= \nu + \bar{c}(2\gamma + \mu) + c\lambda + \bar{c}^2(\tau + 2\beta) + c\bar{c}(\pi + 2\alpha) + \bar{c}^3\sigma + c\bar{c}^2(\rho + 2\epsilon) + \\
&\quad c\bar{c}^3\kappa + \Delta\bar{c} + \bar{c}\delta\bar{c} + c\bar{\delta}\bar{c} + c\bar{c}D\bar{c}.
\end{aligned}
\tag{5.45}
$$

5.5 Appendix B: Generalized Kerr-Schild Transformations

Einstein's equations simplify considerably when one considers algebraically special metrics, that is metrics for which the Weyl tensor has repeated principal null directions (page 63). An important example of these are the Kerr-Schild metrics of the form $\eta_{\mu\nu} + k_\mu k_\nu$ where $\eta_{\mu\nu} = (-1, 1, 1, 1)$ is the Minkowski metric and k^μ is a null vector in flat spacetime [17].

A spacetime with metric tensor $\hat{g}_{\mu\nu}$ is said to be a generalized Kerr-Schild spacetime $\hat{V}$ when ([60], [58])

$$
g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = g_{\mu\nu} + k_\mu k_\nu \tag{5.46}
$$

where g is the metric of an arbitrary spacetime V and k is a null, geodesic, shear-free vector field in V. If V is Minkowski spacetime with the flat metric $\eta_{\mu\nu}$ then $\hat{V}$ is referred to as a special Kerr-Schild spacetime (if one drops the requirement that k be geodesic and shear-free then $\hat{V}$ is simply a Kerr-Schild spacetime [57]). To avoid having to talk about "special generalized Kerr-Schild metrics" we will assume in this thesis that a generalized Kerr-Schild transformation has k shear-free and geodesic (this is not always the case in the literature c.f. [58], [60]).

Clearly k remains null in $\hat{V}$. Let $\{k, l, m, \bar{m}\}$ be a null tetrad with respect to g; the corresponding tetrad with respect to $\hat{g}$ is $\{\hat{k}^\mu = k^\mu, \hat{l}^\mu = l^\mu - (1/2)k^\mu, \hat{m}^\mu = m^\mu, \hat{\bar{m}}^\mu = \bar{m}^\mu\}$. One can show that the spin-coefficients of this tetrad transform as (see [60])

$$
\begin{aligned}
\hat{\kappa} &= \kappa, \\
\hat{\sigma} &= \sigma, \\
\hat{\rho} &= \rho, \\
\hat{\pi} &= \pi, \\
\hat{\epsilon} &= \epsilon, \\
\hat{\tau} &= \tau, \\
\hat{\alpha} &= \alpha + \frac{1}{4}\kappa, \\
\hat{\beta} &= \beta + \frac{1}{4}\bar{\kappa}, \\
\hat{\lambda} &= \lambda + \frac{1}{2}\bar{\sigma}, \\
\hat{\mu} &= \mu + \frac{1}{2}\rho, \\
\hat{\gamma} &= \gamma + \frac{1}{2}\bar{\epsilon} + \frac{1}{4}(\rho - \bar{\rho}), \\
\hat{\nu} &= \nu - \frac{1}{2}(\pi + \bar{\tau}) + \alpha + \bar{\beta} + \frac{1}{4}\bar{\kappa}.
\end{aligned}
$$

Thus, we see that provided k is geodesic and shear-free in V, then it is geodesic and shear-free in $\hat{V}$ also.

Chapter 6

Conclusion

In this dissertation we have considered the physical interaction of a cosmic string with a black hole. In particular, we have looked at stationary configurations of an very long string that has been trapped by the black hole. We began by writing down the equations of motion for a rigidly rotating string in the Kerr-Newman spacetime. These were solved in the non-rotating case (where $\Omega = 0$) and in for all Ω in the equtorial plane. We were able to classify these rigidly rotating solutions into "sub-luminal" and "super-luminal" cases, depending on whether the corresponding Killing vector was timelike or spacelike. Of particular interest were the strings with end-points on the horizon of the black hole; these may prove interesting candidates for sources of gravitational radiation, as our preliminary investigations indicate.

We also studied the stationary strings in detail. In particular, we studied the cone strings which were shown to be the only regular, stationary Nambu-Goto string solutions which pass through the static limit of a Kerr-Newman black hole. These strings have the remarkable property that the induced geometry of the world-sheet is that of a two-dimensional black hole with a horizon located at the intersection of the static limit and the world sheet. It is interesting that a physical manifestation of a two-dimensional black hole may exist. Further, it was shown that signals could propagate from the interior of the two-dimensional black hole to the exterior, via the extra dimensions of the four-dimensional spacetime. This may have interesting implications in the study of information loss from four-dimensional black holes.

The theorem that demonstrates that these principal Killing surfaces are the only regular stationary world sheets that pass through the static limit is a new kind of uniqueness theorem for black holes. It also shows that these cone string configuarions are the only stationary configurations possible for a very long, trapped cosmic string.

It appears unlikely that a similar theorem holds for the rigidly rotating strings (with $\Omega \neq 0$) however.

These distinguished cone string world sheets were shown each to be tangent to a principal null direction of the Weyl tensor in the Kerr-Newman spacetime. We were able to find minimal surfaces in a class of algebraically special vacuum spacetimes, which are also tangent to a principal null direction of the Weyl tensor. In Petrov type-D spacetimes, we discussed when these strings were stationary.

We also looked at these minimal surfaces in Kerr-Schild spacetimes and showed that their properties could be understood by examining corresponding minimal surfaces in flat spacetime. The technique of using a orthonormal tetrad for describing the geometry of two-surfaces is not new. However, we have shown that it can be particularly useful in describing minimal surfaces and may prove to have other applications in the future.

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